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DIVIDENDS AND COMPOUND POISSON PROCESSES: A NEW STOCHASTIC STOCK PRICE MODEL

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This study introduces a stochastic multi-period dividend discount model (DDM) that includes (i) a compound nonhomogenous Poisson process for dividend growth and (ii) the probability of firm default. We obtain maximum likelihood (ML) estimators and confidence interval formulas of our model parameters. We apply the model to a set of firms from the S&P 500 index using historical dividend and price data over a 42-year period. Interestingly, stock price estimations calculated with the model are close to the observable prices. Overall, we prove that the model can be a useful tool for stock pricing.

Keywords: Stochastic dividend discount model; compound nonhomogeneous poisson process; random time of firm default; ML estimators.

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1. Introduction

This study consists of two parts. First, we build an enhanced model for stock price valuation — a stochastic dividend discount model (DDM) based on compound nonhomogeneous Poisson processes. Second, we empirically apply the model to the market data of US firms from the S&P 500 index to show the model's validity.

DDMs, first introduced by Williams (1938), are a popular tool for stock valuation. The basic idea is that the market price of a stock equals the present value of the future dividends paid by the firm, Jawadi & Prat (2017). By their very nature, DDM approaches are best applicable to companies paying regular cash dividends. If a company does not do so, other stock valuation approaches are preferable, e.g. Nawalkha *et al.* (2011). In spite of this limitation, DDMs have evolved into a useful stock valuation approach — comparable to pendants such as duration measures, which are popular in straight bond, but less helpful in zerobond valuation. The two crucial research fields of DDMs are (i) the modeling of future dividend payments and (ii) future defaults of the respective firm, see Lazzati & Menichini (2015). This paper aims to contribute to both fields with an enhanced stochastic DDM: (i) It introduces multi-period conditional and unconditional moments of future stock prices. (ii) It includes (at a random point in time) the possibility of corporate default. Because neither of these domains had been explored in depth so far, the novelty of our paper on stock valuation is twofold.

This paper is organized as follows. Section 2 provides an overview of the current literature on DDMs, with the remaining shortfalls identified. In Sec. 3, we introduce a stochastic DDM with a compound nonhomogeneous Poisson process, and we obtain maximum likelihood (ML) estimators of our model. Section 4 is devoted to a numerical analysis of the stochastic DDM, i.e. we apply it to real market data, namely stocks belonging to the S&P500. Section 5 concludes.

2. Literature Review

As the outcome of DDMs depends crucially on dividend forecasts, most research in the last few decades has been around the proper estimations of dividend development: After Williams (1938) initial model, Gordon & Shapiro (1956) introduced a more sophisticated one, in what dividends change by a defined (deterministic) growth rate. In the wake of this "Gordon Growth Model", three dividend phases (extraordinary, incremental, and moderate growth) are introduced to those deterministic DDMs to obtain more realistic valuation results, see Molodovsky *et al.* (1965), and Sorensen & Williamson (1985).

A log version of the first stochastic DDM was then introduced by Campbell & Shiller (1988), who derived a connection between price, dividend, and return by approximation. Hurley & Johnson (1994) further enhanced the deterministic DDM by first assuming that the growth rate of dividends is described by a Bernoulli random variable. Their model considers the default probability of the firm for firms whose are priced. Second, they extended their model to a two-phase DDM, see Hurley & Johnson (1997). Hurley & Johnson (1994) model was then enhanced by Yao (1997) with a trinomial evolution for future increases in dividends. In turn, Hurley & Johnson (1998) extended Yao's model with a dividend growth rate as a discrete random variable. Hurley (2013) considered the growth rate of the stochastic dividend as a continuous random variable with a given density function. Further, Agosto & Moretto (2015) proposed a discrete distribution of the dividend growth rate and used a closed-form expression for the variance of the stock price.

So far, aforementioned papers have investigated growth rates as an identically distributed random variable. Identically distributed random variables, however, are not a realistic assumption when modeling dividend payments. Instead, growth rates usually follow different distribution patterns as companies pass through different periods of the corporate life cycle. Therefore, to avoid this shortcoming of previous research and to fill the respective research gap, we modeled the expected future dividends by a stochastic process, namely a compound nonhomogeneous Poisson-process, because it considers that stock prices may jump, see Cox & Ross (1976). It should be noted that a nonhomogeneous Poisson-process is a special case of the Cox process, see, e.g. Mikosch (2009) and Lando (1998). In financial economics, compound nonhomogeneous Poisson-processes have been used to model a total claim amount in nonlife insurance, see Mikosch (2009), furthermore, the usage of nonhomogeneous Poisson-processes can be found in option pricing (Cont & Tankov 2004), bond pricing with default risk (Lando 1998), modeling of high-frequency transaction data (Rydberg & Shephard 2003), and bid arrival times in online auctions (Shmueli *et al.* 2004).

Therefore, the model we propose provides a multi-period stock price, see Sec. 3.1. Furthermore, our paper departs from previous studies that focused on a zero-period stock price only. Additionally, we also include a random time of default for the firm, see Sec. 3.2, and we provide a method to calculate moments of future random stock prices, see Sec. 3.3.

3. Methodology: Compound Nonhomogeneous Poisson Process

Let us introduce the following notation, dealing with a compound nonhomogeneous Poisson process. For any real-value function f on $[0, \infty)$, we write

$$f(s,t] := f(t) - f(s), \quad 0 \le s \le t < \infty. \tag{3.1}$$

Table 1 summarizes the most relevant variables used in this paper.

Let $N(t)$ be a nonhomogeneous Poisson process with the mean value function $\mu(t)$ (see Mikosch (2009)), that is, its distribution function is given by

$$\mathbb{P}(N(t) = n) = \frac{(\mu(t))^n}{n!} e^{-\mu(t)} \quad \text{for } n = 0, 1, 2, \ldots \tag{3.2}$$

Table 1. Definition of variables.

Notation	Description
$N(t)$	Nonhomogeneous Poisson process
$\mu(t)$	Mean function of nonhomogeneous Poisson process
$n+1$	Future periods
$Y_i^{(j)}$	For the jth period, the ith random jump size of the dividend
t_j	Partition of time
$Q(t)$	Compound nonhomogeneous Poisson process at time t
d_t	Dividend amount at time t
$Q^{(j)}(s,t]$	For the jth period, the dividend increment from time s to time t
τ	Random time of company default
k	Required rate of return of company's stock
P_t	Random stock price at time t
p	Moment order of random stock price
α_p	pth moment of random stock price
λ	Intensity of homogeneous Poisson process, i.e. $\mu(t) = \lambda t$

with $\mu(t) = \mathbb{E}(N(t))$, where $\mathbb{P}$ is a probability measure and $\mathbb{E}$ is an expectation operator, and let $Y_1^{(j)}, Y_2^{(j)}, \ldots,\ j = 1, \ldots, n+1$ be a sequence of identically distributed random variables. We assume that the random variables $Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_1^{(n+1)}, Y_2^{(n+1)}, \ldots$ are independent of one another and also independent of the Poisson process $N(t)$. Suppose that the future time frame consists of $n+1$ periods, then let $t_0 = 0 < t_1 < t_2 < \cdots < t_n < t_{n+1} := \infty$ be a partition of the time. Subsequently, the first period covers 0 to t_1, the second period covers $t_1 + 1$ to t_2 and so on, such that the $(n+1)$th period covers $t_n + 1$ to infinity.

3.1. *Dividends*

Dividend payments can follow various patterns, especially because companies pass through different levels of economic success during their corporate life cycles, during which various reasons for an increasing, decreasing or even ceasing of dividend payments apply. Therefore, the effects of dividend payments on stock prices have been extensively studied (Acker 1999, Best & Best 2001, Che & Fuller 2020, Huang *et al.* 2017, Jakob & Whitby 2017, Twu 2010). If we graphically analyze dividend payments of firms, two characteristics become obvious: (1) dividends are randomly constant over periods, and (2) dividend level jumps are random, too, as most companies prefer to smooth dividends to signal continuity in the short run, and alter dividends only in the event of significant changes which promise another sustainable dividend level (seminal, see Lintner (1956), of those based hereupon, see, e.g. Baker *et al.* (1985), Baker & Powell (1999), and Bhattacharyya (2007)). Figure 1 illustrates this pattern for a selected stock. In this figure, we consider the 1999–2019 dividend history of First Energy (FE), an electric utility company included in the S&P 500 index. Comparable patterns can be found for numerous stocks (e.g. Kao *et al.* (1991) and van Hilten *et al.* (1993)). Altogether, it is reasonable to assume that future dividends will follow a compound nonhomogeneous Poisson process.

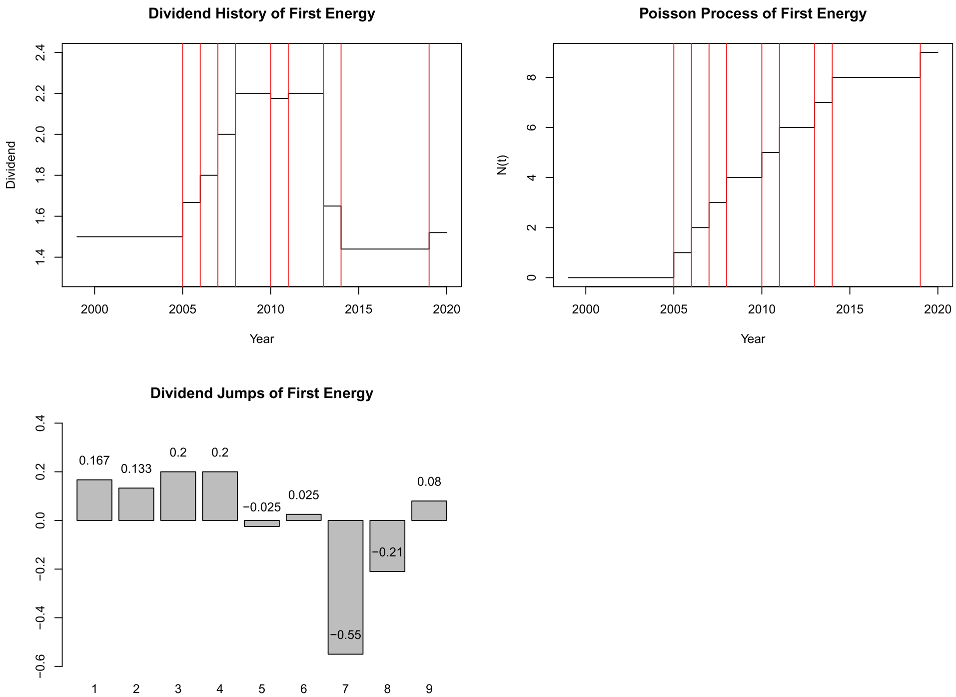

Fig. 1. Dividend history of FE.

We introduce the following compound nonhomogeneous Poisson process in the jth period:

$$Q^{(j)}(t) := \sum_{i=1}^{N(t)} Y_i^{(j)}, \quad j = 1, 2, \dots, n+1. \tag{3.3}$$

The jumps in $Q^{(j)}(t)$ occur at the same time as the jumps in $N(t)$, but the jumps in $N(t)$ always equal 1 whereas the jumps in $Q^{(j)}(t)$ are random. The first jump is of the size $Y_1^{(j)}$ and the second of $Y_2^{(j)}$, and so on. We fix t, such that $t_{j-1} < t \leq t_j$ throughout the paper. In the jth period, we model dividend increment from time s to time t by following a compound nonhomogeneous Poisson process

$$d_t - d_s = Q^{(j)}(s, t] = \sum_{i=N(s)+1}^{N(t)} Y_i^{(j)}, \tag{3.4}$$

where for $j = 1, \dots, (n+1)$, $a, b \in \mathbb{Z}_+$ and $a < b$, we assume $\sum_{i=b}^{a} Y_i^{(j)} = 0$. Then, the dividend payment at time t is given by the following equation:

$$d_t = d_0 + Q^{(1)}(t_1) + Q^{(2)}(t_1, t_2] + \cdots + Q^{(j-1)}(t_{j-2}, t_{j-1}] + Q^{(j)}(t_{j-1}, t]$$

$$= d_0 + \sum_{i=1}^{N(t_1)} Y_i^{(1)} + \sum_{i=N(t_1)+1}^{N(t_2)} Y_i^{(2)}$$

$$+ \cdots + \sum_{i=N(t_{j-2})+1}^{N(t_{j-1})} Y_i^{(j-1)} + \sum_{i=N(t_{j-1})+1}^{N(t)} Y_i^{(j)}. \tag{3.5}$$

This means that the dividend patterns will be modeled by the provided formula.

3.2. *Random stock price*

In this section, we model the random stock price, for which we derive conditional as well as unconditional distribution functions. Since the original distribution functions are "unhandy" to use — to approximate the distributions — we develop a moments formula for the random stock price. To this end, let $\tau \in \mathbb{N}$ be the random time of company default and independent of random variables $Y_1^{(1)}, Y_2^{(1)}, \dots, Y_1^{(n+1)}, Y_2^{(n+1)}, \dots$ and the Poisson process $N(t)$. Default events of a company are generally associated with financial characteristics such as inadequate cash flow to service debt, declining revenues or operating margins, high leverage, declining or marginal liquidity, and the inability to successfully implement a business plan. Whereas respective information can be confidential, or remain unrevealed until the next disclosure date, the probabilities of default of numerous firms are assessed by external rating agencies such as Standard and Poor's, Fitch or Moody's Investors Service, whose credit ratings are accessible for the various stakeholders of the firm. Also, we define the following notations:

$$F_t(s) := \frac{1}{k(1+k)^{s-t-1}}, \tag{3.6}$$

where k is the required rate of return of common stock. To obtain the (random) price of the stock, we assume that before time t, a default does not occur and the firm will default after time t in the ith ($i \geq j$) period (that is, $t_{i-1} < \tau \leq t_i$). Note that the following inequality holds for the times: if default occurs in the jth period, then $t_{j-1} < t < \tau \leq t_j$ and if default occurs in ith ($i > j$) period, $t_{j-1} < t \leq t_j \leq t_{i-1} < \tau \leq t_i$. Based hereupon, we obtain the subsequent result for the random price of a common stock.

Proposition 3.1. *If a default occurs in the* i*th* $(i \geq j)$ *period* (*that is,* $t_{i-1} < \tau \leq t_i$), *then the random price of common stock at time* t *is given by*

$$P_t = F_t(\tau, t+1]d_t + \sum_{m=t+1}^{t_j} F_t(\tau, m]Q^{(j)}(m-1, m]$$

$$+ \sum_{m=t_j+1}^{t_{j+1}} F_t(\tau, m] Q^{(j+1)}(m-1, m]$$

$$+ \cdots + \sum_{m=t_{i-1} \vee t+1}^{\tau-1} F_t(\tau, m] Q^{(i)}(m-1, m], \tag{3.7}$$

where $a \vee b = \max(a, b)$.

Proof of any proposition presented can be found in the appendix of this paper.

3.2.1. *Distribution functions*

As the next step, we develop a distribution function for the random stock price. This is necessary because it allows us to derive the theoretical price and the price confidence band of a stock. Therefore, we introduced the following notations: for $a \leq b$, $a, b \in \mathbb{Z}_+$ and $\ell = j+1, \ldots, i$,

$$Q_s^{(\ell)}(z_a : z_b] := \sum_{m=a+1}^{b} F_t(s, m] \sum_{i=z_{m-1}+1}^{z_m} Y_i^{(\ell)}, \tag{3.8}$$

$$\begin{aligned} &R_s[z_{t_1}, z_{t_2}, \ldots, z_{t_{j-1}}, z_t]] \\ &\quad := F_t(s, t+1] \left\{ d_0 + \sum_{i=1}^{z_{t_1}} Y_i^{(1)} + \sum_{i=z_{t_1}+1}^{z_{t_2}} Y_i^{(2)} + \cdots \right. \\ &\qquad \left. + \sum_{i=z_{t_{j-2}}+1}^{z_{t_{j-1}}} Y_i^{(j-1)} + \sum_{i=z_{t_{j-1}}+1}^{z_t} Y_i^{(j)} \right\}. \end{aligned} \tag{3.9}$$

Based on this, we have the following result for the conditional and unconditional distribution function of the random stock price.

Proposition 3.2. *Let P_t be a random stock price at time t. Then the conditional distribution function of the random stock price is given by*

$$\begin{aligned} F_{P_t \mid \tau}(x \mid t) &= \mathbb{P}(P_t \leq x \mid \tau > t) \\ &= \sum_{s=t+1}^{\infty} \left\{ \sum_{z_{t_1}=0,\ldots,z_{t_j}=z_{t_{j-1}}, z_{t+1}=z_t,\ldots,z_{s-1}=z_{s-2}} \mathbb{P}(R_s[z_{t_1}, z_{t_2}, \ldots, z_{t_{j-1}}, z_t] \right. \\ &\quad + Q_s^{(j)}(z_t : z_{t_j}] + Q_s^{(j+1)}(z_{t_j} : z_{t_{j+1}}] + \cdots \\ &\quad + Q_s^{(i)}(z_{t_{i-1} \vee t} : z_{s-1}] \leq x) e^{-\mu(s-1)} \times \frac{\mu(t_1)^{z_{t_1}}}{z_{t_1}!} \end{aligned}$$

$$\times \frac{\mu(t_1,t_2]^{z_{t_2}-z_{t_1}}}{(z_{t_2}-z_{t_1})!}\cdots\frac{\mu(t_{j-1},t]^{z_t-z_{t_{j-1}}}}{(z_t-z_{t_{j-1}})!}\prod_{m=t+1}^{s-1}\frac{\mu(m-1,m]^{z_m-z_{m-1}}}{(z_m-z_{m-1})!}\Bigg\}$$
$$\times\,\mathbb{P}(\tau=s\,|\,\tau>t) \tag{3.10}$$

and the unconditional distribution function of the random stock price is given by

$$F_{P_t}(x)=\mathbb{P}(P_t\le x)$$
$$=\lim_{s\to\infty}\Bigg\{\sum_{z_{t_1}=0,\dots,z_{t_j}=z_{t_{j-1}},z_{t+1}=z_t,\dots,z_s=z_{s-1}}\mathbb{P}(R_\infty[z_{t_1},z_{t_2},\dots,z_{t_{j-1}},z_t]$$
$$+Q_\infty^{(j)}(z_t:z_{t_j}]+Q_\infty^{(j+1)}(z_{t_j}:z_{t_{j+1}}]+\cdots+Q_\infty^{(n+1)}(z_{t_n}:z_s]\le x)e^{-\mu(s)}$$
$$\times\frac{\mu(t_1)^{z_{t_1}}}{z_{t_1}!}\frac{\mu(t_1,t_2]^{z_{t_2}-z_{t_1}}}{(z_{t_2}-z_{t_1})!}\cdots\frac{\mu(t_{j-1},t]^{z_t-z_{t_{j-1}}}}{(z_t-z_{t_{j-1}})!}$$
$$\times\prod_{m=t+1}^{s}\frac{\mu(m-1,m]^{z_m-z_{m-1}}}{(z_m-z_{m-1})!}\Bigg\}. \tag{3.11}$$

From the above exact distributions, we can see that it is almost impossible to use the precise distributions of a random price of a common stock P_t. If all moments are finite and the power series based on the moments are defined on an open interval that contains zero, then the moments uniquely define a distribution function, see Billingsley (1995). If at least a part of the moments is finite, we can approximate the distribution function of P_t by tools based on moments, see, e.g. Provost (2005). Thus, we need to calculate moments of the future random price of the common stock.

Here we provide the main results of the approximation of the distribution function following Provost (2005). Let $Y_1^{(1)},Y_2^{(1)},\dots,Y_1^{(n+1)},Y_2^{(n+1)}$ be continuous random variables. Then, P_t is a continuous random variable. We denote the density function of a generic random variable X by f_X. We assume further that an interval (a,b) covers the random values of P_t with sufficiently high probability, that is, $\mathbb{P}(a<P_t<b)=1-\varepsilon$, where $\varepsilon>0$ is a sufficiently small number.

Next we consider the approximation method for the unconditional density function of the random stock price at time t, $f_{P_t}(x)$, since the principle of approximation is the same for a conditional density function. Let $Q:=(P_t-u)/s$, where $u\in\mathbb{R}$ and $s>0$, $a_0:=(a-u)/s$, $b_0:=(b-u)/s$, for $p=1,\dots,n$, $\delta_p:=\mathbb{E}(Q^p)$, $\psi_Q(x)=cw(x)$, where $c>0$ is a normalizing constant and w is a weight function, be an initial density approximation of $f_Q(x)$ and for $p=0,1,\dots,2n$, $\bar{\delta}_p:=\int_{a_0}^{b_0}x^p\psi_Q(x)dx$. We suppose that for $p=0,1,2,\dots$, δ_p uniquely define the distribution of Q, that for $p=0,1,\dots,2n$, $\bar{\delta}_p$ exist, and that whenever $\psi_Q(x)$ is a nontrivial function of x, its tail behavior is congruent to that of $f_Q(x)$. Let M be an $(n+1)\times(n+1)$ matrix, and for $p = 0, 1, \dots, n$, its $(p+1)$th row be $(\bar{\delta}_p, \dots, \bar{\delta}_{p+n})$. In this case, for the density function $f_{P_t}(x)$, the following approximation holds:

$$f_{P_t}(x) \approx \psi_Q\left(\frac{x-u}{s}\right)\sum_{m=0}^{n}\frac{\xi_m}{s}\left(\frac{x-u}{s}\right)^m, \tag{3.12}$$

where $(\xi_0, \dots, \xi_n)^T = M^{-1}(\delta_0, \dots, \delta_n)^T$ and ξ_m, $m = 0, 1, \dots, n$ are determined by the method of moments. Now we list the parameters, which correspond to equation (3.12) of Laguerre, Legendre, Jacobi and Hermite polynomials see Table 2. For Table 2, $B(x, y)$ is the Beta function, $\Gamma(x)$ is the Gamma function, and, finally,

$$\begin{aligned} &v := \frac{(\mathbb{E}(P_t) - a)^2}{\mathbb{E}(P_t^2) - \mathbb{E}^2(P_t)} - 1, \quad \alpha := \mathbb{E}(P_t)\frac{\mathbb{E}(P_t) - \mathbb{E}(P_t^2)}{\mathbb{E}(P_t^2) - \mathbb{E}^2(P_t)} - 1, \\ &\beta := (\alpha+1)\frac{1 - \mathbb{E}(P_t)}{\mathbb{E}(P_t)} - 1, \end{aligned} \tag{3.13}$$

where for $p = 1, 2$, $\mathbb{E}(P_t^p)$ is calculated by Eq. (3.20), see below.

To illustrate usage of Provost's (2005) approximation for our model, we focus on the Laguerre approximant, since an idea is same for the other approximants and conditional density function. For the Laguerre approximant, the matrix M and the vector $\delta := (\delta_1, \dots, \delta_n)^T$ are given by

$$M = \begin{bmatrix} \frac{\Gamma(v+1)}{\Gamma(v+1)} & \frac{\Gamma(v+2)}{\Gamma(v+1)} & \cdots & \frac{\Gamma(v+1+n)}{\Gamma(v+1)} \\ \frac{\Gamma(v+2)}{\Gamma(v+1)} & \frac{\Gamma(v+3)}{\Gamma(v+1)} & \cdots & \frac{\Gamma(v+2+n)}{\Gamma(v+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(v+1+n)}{\Gamma(v+1)} & \frac{\Gamma(v+2+n)}{\Gamma(v+1)} & \ddots & \frac{\Gamma(v+1+2n)}{\Gamma(v+1)} \end{bmatrix},$$

$$\delta = \begin{bmatrix} 1 \\ \frac{1}{s^1}\sum_{p=0}^{1} C_1^p(-1)^{1-p}\mathbb{E}(P_t^p)a^{1-p} \\ \vdots \\ \frac{1}{s^n}\sum_{p=0}^{n} C_n^p(-1)^{n-p}\mathbb{E}(P_t^p)a^{n-p} \end{bmatrix},$$

where for $p = 0, \dots, n$, $C_n^p = \frac{n!}{p!(n-p)!}$ is the binomial coefficient, $s = \frac{\mathbb{E}(P_t^2) - \mathbb{E}^2(P_t)}{\mathbb{E}(P_t) - a}$ is the scale factor, and $\mathbb{E}(P_t^p)$ is calculated by Eq. (3.20), see below. As a result, the

Table 2. Parameters of approximants.

	Laguerre	Legendre	Jacobi	Hermite
u	a	$\frac{a+b}{2}$	a	δ_1
s	$\frac{\mathbb{E}(P_t^2)-\mathbb{E}^2(P_t)}{\mathbb{E}(P_t)-a}$	$\frac{b-a}{2}$	$b-a$	$\sqrt{\mathbb{E}(P_t^2)-\mathbb{E}^2(P_t)}$
$\psi_Q(x)$	$\frac{x^v e^{-x}}{\Gamma(v+1)},\ x>0$	$\frac{1}{2},\ x\in[-1,1]$	$\frac{x^\alpha(1-x)^\beta}{B(\alpha+1,\beta+1)},\ x\in[0,1]$	$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, x\in\mathbb{R}$
$\bar{\delta}_p, p=0,1,\ldots$	$\frac{\Gamma(v+1+p)}{\Gamma(v+1)}$	$\frac{1-(-1)^{p+1}}{2(p+1)}$	$\frac{\Gamma(\alpha+\beta+2)\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)\Gamma(\alpha+p+\beta+2)}$	$\frac{2^{(p-1)/2}(1+(-1)^p)\Gamma(\frac{1+p}{2})}{\sqrt{2\pi}}$

density function of the price at time t is approximated by the following equation:

$$f_{P_t}(x) \approx \frac{e^{-(x-a)/s}}{\Gamma(v+1)} \sum_{m=0}^{n} \frac{\xi_m}{s} \left(\frac{x-a}{s}\right)^{v+m}, \quad a \le x \le b,$$

where $(\xi_0, \ldots, \xi_n)^T = M^{-1}\delta$ and $v = \frac{(\mathbb{E}(P_t)-a)^2}{\mathbb{E}(P_t^2)-\mathbb{E}^2(P_t)} - 1$.

3.2.2. *Moments*

Moments of the random stock price are used not only to approximate the distribution function but also to give information about a random stock price, for example

(1) *expectation* — (first moment) gives a theoretical price,
(2) *standard deviation* — (second central moment) gives a stock price volatility,
(3) *skewness* – (standardized ratio between third and second central moments) gives information if a random stock price is skewed to the right or the left and
(4) *kurtosis* – (standardized ratio between fourth and second central moments) expresses if a stock price appears heavy or light tailed as compared to a normal distribution.

Now we present a way to calculate moments of future random stock prices. Let $\phi_{Y^{(j)}}(u) := \mathbb{E}(e^{iuY_1^{(j)}})$, for $j = 1, \ldots, n+1$, where i is an imaginary unit. Then, for $t \le s$, the characteristic function of a random variable $Q^{(j)}(t,s]$ is given by

$$\phi_{Q^{(j)}(t,s]}(u) = \mathbb{E}(e^{iuQ^{(j)}(t,s]}) = \exp\{\mu(t,s](\phi_{Y_1^{(j)}}(u) - 1)\}, \tag{3.14}$$

where we use the well-known formula for a characteristic function of a compound nonhomogeneous Poisson process, see Mikosch (2009). The random variables $Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_1^{(n+1)}, Y_2^{(n+1)}, \ldots$ are independent of one another and also independent of the Poisson process $N(t)$, so for nonoverlapping intervals, $Q^{(1)}$, $Q^{(2)}$, …, $Q^{(i)}$ are also independent random variables. Then, the conditional characteristic function of the random price of a stock at time t, that is, P_t conditional on $\tau = s$, is given by

$$\mathbb{E}(e^{iuP_t} \mid \tau = s) = \exp\{h_t(u \mid s)\}, \tag{3.15}$$

where

$$\begin{aligned} h_t(u\,|\,s) := {} & iuF_t(s,t+1]d_0 + \mu(t_1)(\phi_{Y_1^{(1)}}(uF_t(s,t+1]) - 1) + \cdots \\ & + \mu(t_{j-2},t_{j-1}](\phi_{Y_1^{(j-1)}}(uF_t(s,t+1]) - 1) \\ & + \mu(t_{j-1},t](\phi_{Y_1^{(j)}}(uF_t(s,t+1]) - 1) \\ & + \sum_{m=t+1}^{t_j} \mu(m-1,m](\phi_{Y_1^{(j)}}(uF_t(s,m]) - 1) \\ & + \sum_{m=t_j+1}^{t_{j+1}} \mu(m-1,m](\phi_{Y_1^{(j+1)}}(uF_t(s,m]) - 1) + \cdots \\ & + \sum_{m=t_{i-1}\vee t+1}^{s-1} \mu(m-1,m](\phi_{Y_1^{(i)}}(uF_t(s,m]) - 1). \end{aligned} \tag{3.16}$$

Let $h_t(u\,|\,s)$ be a sufficiently differentiable function for all $s > t$. We also assume that for a fixed p, the random variables $Y_1^{(j)p}, j = 1,\dots,n+1$ are integrable. Then, for $i = 1,\dots,p-1$, $Y_1^{(j)i}, j = 1,\dots,n+1$ the random variables are integrable, too, see Billingsley (1995). If we use the well-known result of a characteristic function, that is, $\phi^{(p)}_{Y_1^{(j)}}(0) = i^p\mathbb{E}(Y_1^{(j)p})$, the pth derivative of the function $h_t(u\,|\,s)$ at a point $u = 0$ becomes

$$\begin{aligned} h_t^{(p)}(0\,|\,s) = i^p \Bigg\{ & d_0 I_{\{1\}}(p)F_t(s,t+1] + \mathbb{E}(Y_1^{(1)p})\mu(t_1)F_t(s,t+1]^p + \cdots \\ & + \mathbb{E}(Y_1^{(j-1)p})\mu(t_{j-2},t_{j-1}]F_t(s,t+1]^p + \mathbb{E}(Y_1^{(j)p})\mu(t_{j-1},t]F_t(s,t+1]^p \\ & + \mathbb{E}(Y_1^{(j)p}) \sum_{m=t+1}^{t_j} \mu(m-1,m]F_t(s,m]^p \\ & + \mathbb{E}(Y_1^{(j+1)p}) \sum_{m=t_j+1}^{t_{j+1}} \mu(m-1,m]F_t(s,m]^p + \cdots \\ & + \mathbb{E}(Y_1^{(i)p}) \sum_{m=t_{i-1}\vee t+1}^{s-1} \mu(m-1,m]F_t(s,m]^p \Bigg\}, \end{aligned} \tag{3.17}$$

where $I_{\{1\}}(p)$ is the indicator function. That is, if $p = 1$, then I equals 1, and 0 otherwise.

If the company continues its business for longer than the tth period, the conditional characteristic function of the random price of stock at time t, which is P_t

conditional on $\tau > t$, is given by

$$\mathbb{E}(e^{iuP_t} \,|\, \tau > t) = \sum_{s=t+1}^{\infty} \exp\{h_t(u \,|\, s)\}\mathbb{P}(\tau = s \,|\, \tau > t). \tag{3.18}$$

If we use Faa di Bruno's formula (see Johnson (2002)), then the pth moment of the price of a stock at time t (conditional on $\tau > t$) is given by

$$\begin{aligned}\mathbb{E}(P_t^p \,|\, \tau > t) &= \frac{1}{i^p} \sum_{s=t+1}^{\infty} \frac{d^p}{du^p}(\exp\{h_t(u \,|\, s)\})\bigg|_{u=0} \mathbb{P}(\tau = s \,|\, \tau > t) \\ &= \frac{1}{i^p} \sum_{s=t+1}^{\infty} \left(\sum \frac{p!}{b_1! \ldots b_p!} \left(\frac{h_t'(0 \,|\, s)}{1!}\right)^{b_1} \cdots \left(\frac{h_t^{(p)}(0 \,|\, s)}{p!}\right)^{b_p}\right) \\ &\quad \times \mathbb{P}(\tau = s \,|\, \tau > t),\end{aligned} \tag{3.19}$$

where the sum is over all different solutions in nonnegative integers $b_1, \ldots, b_p$ of $b_1 + 2b_2 + \cdots + pb_p = p$. Next, we assume that a default never occurs, so that a company will pay dividends forever, that is $P(\tau = \infty) = 1$. In that case, the pth moment of the price of a stock at time t is given by

$$\mathbb{E}(P_t^p) = \frac{1}{i^p} \sum \frac{p!}{b_1! \ldots b_p!} \left(\frac{h_t'(0 \,|\, \infty)}{1!}\right)^{b_1} \cdots \left(\frac{h_t^{(p)}(0 \,|\, \infty)}{p!}\right)^{b_p}, \tag{3.20}$$

where the sum is over all different solutions in nonnegative integers $b_1, \ldots, b_p$ of $b_1 + 2b_2 + \cdots + pb_p = p$ and

$$\begin{aligned}h_t^{(p)}(0 \,|\, \infty) = i^p \Bigg\{ & d_0 I_{\{1\}}(p) F_t(t+1) + \mathbb{E}(Y_1^{(1)p})\mu(t_1)F_t(t+1)^p + \cdots \\ & + \mathbb{E}(Y_1^{(j-1)p})\mu(t_{j-2}, t_{j-1}]F_t(t+1)^p + \mathbb{E}(Y_1^{(j)p})\mu(t_{j-1}, t]F_t(t+1)^p \\ & + \mathbb{E}(Y_1^{(j)p}) \sum_{m=t+1}^{t_j} \mu(m-1, m]F_t(m)^p + \cdots \\ & + \mathbb{E}(Y_1^{(n)p}) \sum_{m=t_{n-1}+1}^{t_n} \mu(m-1, m]F_t(m)^p \\ & + \mathbb{E}(Y_1^{(n+1)p}) \sum_{m=t_n+1}^{\infty} \mu(m-1, m]F_t(m)^p \Bigg\}.\end{aligned} \tag{3.21}$$

Let $\alpha_p = \mathbb{E}(P_t^p \,|\, \tau > t)$ or $\alpha_p = \mathbb{E}(P_t^p)$, $p = 1, 2, \ldots$ be all finite, and let $\sum_{p=1}^{\infty} \alpha_p x^p/p!$ have a positive radius of convergence. Then, the moments uniquely define the distribution function, see Billingsley (1995). In the following section, we finally insert the derived formulas into the DDM.

3.3. *Application to dividend discount models*

In practice, mean, variance, skewness, and kurtosis are considered beneficial statistics, see Agosto & Moretto (2015). Models based on the first four moments are quite common. For example, a practical application would be to check if the market price of a stock follows a normal distribution. For that purpose, we can apply the (Jarque & Bera 1987) test, which is based on skewness and kurtosis. In this case, we need to calculate the third and fourth moments of a firm's random stock price. Therefore, in the following section, we provide formulas that cover the first four moments for a sufficiently differentiable function $h_t(u \mid s)$ — although it is possible to construct formulas of more than the fourth moment. We now apply the presented formulas (see Eqs. (3.19) and (3.20)) in our model to the first four moments. From Eq. (3.19), we derive

$$\mathbb{E}(P_t \mid \tau > t) = \frac{1}{i} \sum_{s=t+1}^{\infty} h_t'(0 \mid s)\mathbb{P}(\tau = s \mid \tau > t), \tag{3.22}$$

$$\mathbb{E}(P_t^2 \mid \tau > t) = \frac{1}{i^2} \sum_{s=t+1}^{\infty} \{(h_t'(0 \mid s))^2 + h_t''(0 \mid s)\}\mathbb{P}(\tau = s \mid \tau > t), \tag{3.23}$$

$$\begin{aligned}\mathbb{E}(P_t^3 \mid \tau > t) = \frac{1}{i^3} \sum_{s=t+1}^{\infty} &\{(h_t'(0 \mid s))^3 + 3h_t'(0 \mid s)h_t''(0 \mid s) + h_t'''(0 \mid s)\} \\ &\times \mathbb{P}(\tau = s \mid \tau > t),\end{aligned} \tag{3.24}$$

$$\begin{aligned}\mathbb{E}(P_t^4 \mid \tau > t) = \frac{1}{i^4} \sum_{s=t+1}^{\infty} &\{(h_t'(0 \mid s))^4 + 6(h_t'(0 \mid s))^2 h_t''(0 \mid s) + 4h_t'(0 \mid s)h_t'''(0 \mid s), \\ &+ 3(h_t''(0 \mid s))^2 + h_t^{(4)}(0 \mid s)\}\mathbb{P}(\tau = s \mid \tau > t).\end{aligned} \tag{3.25}$$

If the firm never defaults in the future, then from Eq. (3.20), the first four moments can be obtained as

$$\mathbb{E}(P_t) = \frac{1}{i} h_t'(0 \mid \infty), \tag{3.26}$$

$$\mathbb{E}(P_t^2) = \frac{1}{i^2}\{(h_t'(0 \mid \infty))^2 + h_t''(0 \mid \infty)\}, \tag{3.27}$$

$$\mathbb{E}(P_t^3) = \frac{1}{i^3}\{(h_t'(0 \mid \infty))^3 + 3h_t'(0 \mid \infty)h_t''(0 \mid \infty) + h_t'''(0 \mid \infty)\}, \tag{3.28}$$

$$\begin{aligned}\mathbb{E}(P_t^4) = \frac{1}{i^4}&\{(h_t'(0 \mid \infty))^4 + 6(h_t'(0 \mid \infty))^2 h_t''(0 \mid \infty) + 4h_t'(0 \mid \infty)h_t'''(0 \mid \infty), \\ &+ 3(h_t''(0 \mid \infty))^2 + h_t^{(4)}(0 \mid \infty)\}.\end{aligned} \tag{3.29}$$

Therefore, the central moment of the random price of a common stock at a time t is given by

$$\mathrm{Var}(P_t) = \mathbb{E}(P_t^2) - \mathbb{E}^2(P_t) = \frac{1}{i^2} h_t''(0 \,|\, \infty), \tag{3.30}$$

$$\mathbb{E}(P_t - \mathbb{E}(P_t))^3 = \mathbb{E}(P_t^3) - 3\mathbb{E}(P_t^2)\mathbb{E}(P_t) + 2\mathbb{E}^3(P_t) = \frac{1}{i^3} h_t'''(0 \,|\, \infty), \tag{3.31}$$

$$\begin{aligned}\mathbb{E}(P_t - \mathbb{E}(P_t))^4 &= \mathbb{E}(P_t^4) - 4\mathbb{E}(P_t^3)\mathbb{E}(P_t) + 6\mathbb{E}(P_t^2)\mathbb{E}^2(P_t) - 3\mathbb{E}^4(P_t) \\ &= 3\mathrm{Var}^2(P_t) + \frac{1}{i^4} h_t^{(4)}(0 \,|\, \infty).\end{aligned} \tag{3.32}$$

Let us now suppose that the future time frame consists of one period that corresponds to $n = 0$ and $\mu(t) = \lambda t,\ t \geq 0$. Then, the compound nonhomogeneous Poisson process model equals to a simple compound Poisson process. In this case, the first moment and the second, third, and fourth central moments of the random price of the stock are defined by

$$\mathbb{E}(P_t) = \frac{1}{k}\left\{d_0 + \lambda\mathbb{E}(Y_1)\left(t + \frac{1+k}{k}\right)\right\}, \tag{3.33}$$

$$\mathrm{Var}(P_t) = \frac{\lambda\mathbb{E}(Y_1^2)}{k^2}\left(t + \frac{(1+k)^2}{(1+k)^2 - 1}\right), \tag{3.34}$$

$$\mathbb{E}(P_t - \mathbb{E}(P_t))^3 = \frac{\lambda\mathbb{E}(Y_1^3)}{k^3}\left(t + \frac{(1+k)^3}{(1+k)^3 - 1}\right), \tag{3.35}$$

$$\begin{aligned}\mathbb{E}(P_t - \mathbb{E}(P_t))^4 &= 3\left(\frac{\lambda\mathbb{E}(Y_1^2)}{k^2}\left(t + \frac{(1+k)^2}{(1+k)^2 - 1}\right)\right)^2 \\ &\quad + \frac{\lambda\mathbb{E}(Y_1^4)}{k^4}\left(t + \frac{(1+k)^4}{(1+k)^4 - 1}\right).\end{aligned} \tag{3.36}$$

From the above first moment formula, one can deduce that when the intensity of the Poisson process is equal to one ($\lambda = 1$), the theoretical price formula of all the previous papers on additive models (see Hurley & Johnson (1994, 1998), Yao (1997), and Hurley (2013)) are special cases of our zero period formula.

Let us consider a special case, which corresponds to $N(t)$ is the homogeneous Poisson process, that is, $\mu(t) = \lambda$, $t \geq 0$. In this case, it follows from Eqs. (3.21) and (3.26) that theoretical price at time t, which will be used in Sec. 4, is given by

$$\begin{aligned}\mathbb{E}(P_t) = \lambda F_t(t+1)\bigg\{&\frac{d_0}{\lambda} + \mathbb{E}(Y_1^{(1)})t_1 + \cdots + \mathbb{E}(Y_1^{(j-1)})(t_{j-1} - t_{j-2}) \\ &+ \mathbb{E}(Y_1^{(j)})(t - t_{j-1}) + \mathbb{E}(Y_1^{(j)})F_t(t, t_j] + \mathbb{E}(Y_1^{(j+1)})F_t(t_j, t_{j+1}] + \cdots \\ &+ \mathbb{E}(Y_1^{(n+1)})F_t(t_n, \infty]\bigg\}.\end{aligned} \tag{3.37}$$

3.4. *MLE of parameters*

The basic idea of all DDMs is that the market price of a stock equals the sum of the stock's future dividends discounted at a risk-adjusted rate (required rate of return). Therefore, the required rate of return is the main input parameter for DDMs. The most common model to estimate the required rate of return is the capital asset pricing model (CAPM). Using the CAPM is common in practice, but it is a one-factor model (β only) for which criticism applies, see, e.g. Nagorniak (1985). Thus, multi-factor models (e.g. Fama & French (1993)) are therefore often preferred instead. In this paper, for the first time we estimate parameters of the model (including the required rate of return) based on the ML estimation method.

In order to obtain ML estimators of our model, we assume that we have $T+1$ observations of price and dividend data and jump information. That is, our data is $\{d_0, P_0, N(0), \ldots, d_T, P_T, N(T)\}$, where for $t = 0, \ldots, T$, d_t is the dividend observed at time t, P_t is the price observed at time t, and $N(t)$ is the jump information at time t which expresses the number of jumps of dividend up to time t. In order to obtain a parameter estimation, we suppose that at time t, the stock price differs from its theoretical value by a random amount, say u_t, and a firm passes through $r \geq 1$ periods of its corporate life cycle up to time T. For simplicity, we assume that $t_r = T$. In this case, our model becomes

$$\begin{cases} P_t = (1+k)P_{t-1} - d_t + u_t, \\ d_t = d_{t-1} + Q^{(j)}(t-1, t], \\ Q^{(j)}(t-1, t] = \displaystyle\sum_{i=N(t-1)+1}^{N(t)} Y_i^{(j)} \end{cases}, \quad j = 1, \ldots, r, \;\; t_{j-1} < t \leq t_j. \tag{3.38}$$

Now we define

$$i_s := \min\{t \in \{1, \ldots, T\} \,|\, N(t) > N(i_{s-1})\}, \quad s = 1, \ldots, N(T), \tag{3.39}$$

where $i_0 = 0$. Then $\mathcal{I}^{(j)} := \{i_{N(t_{j-1})+1}, \ldots, i_{N(t_j)}\}$, $j = 1, \ldots, r$ sets express jump times of the $N(t)$ process from time t_{j-1} up to time t_j. If we assume that u_t, $t = 1, \ldots, T$ are independent of one another and also independent of Poisson process $N(t)$ and dividend jumps $Y_i^{(j)}$, and

$$\left(\begin{bmatrix} u_t \\ Y_i^{(j)} \end{bmatrix} \sim \mathcal{N} \begin{bmatrix} 0 \\ \mu_j \end{bmatrix}, \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_j^2 \end{bmatrix}\right), \quad j = 1, \ldots, r, \;\; N(t_{j-1}) < i \leq N(t_j), \tag{3.40}$$

where $\mathcal{N}(\tilde{\mu}, \tilde{\Sigma})$ represents a bivariate-normal distribution with a mean of $\tilde{\mu}$ and a covariance matrix of $\tilde{\Sigma}$, and μ_j and σ_j^2 are mean and variance of the jump random variables of the jth period $Y_i^{(j)}$, then it can be shown that the log-likelihood function can be denoted as follows:

$$\begin{aligned}\mathcal{L}(\theta) = &-\frac{N(T)+T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t\in\mathcal{I}}\ln(N(t-1,t]) - \frac{1}{2}\sum_{j=1}^{r}N(t_{j-1},t_j]\ln(\sigma_j^2)\\ &-\frac{1}{2}\sum_{j=1}^{r}\left[\frac{1}{\sigma_j^2}\sum_{t\in\mathcal{I}^{(j)}}\frac{1}{N(t-1,t]}(d_t-d_{t-1}-N(t-1,t]\mu_j)^2\right]\\ &-\frac{T}{2}\ln(\sigma_u^2) - \frac{1}{2\sigma_u^2}\sum_{t=1}^{T}(P_t-(1+k)P_{t-1}+d_t)^2 - \sum_{t=1}^{T}\ln(N(t-1,t]!)\\ &+\sum_{t=1}^{T}N(t-1,t]\ln(\mu(t-1,t]) - \mu(T), \end{aligned}\tag{3.41}$$

where $\mathcal{I} := \bigcup_{j=1}^{r}\mathcal{I}^{(j)}$ and $\theta = (k,\sigma_u^2,\lambda,\mu_1,\dots,\mu_r,\sigma_1^2,\dots,\sigma_r^2)^T$ is the parameter vector. If we define

$$p_{-1} = (P_0,\dots,P_{T-1})^T,\quad p = (P_1,\dots,P_T)^T,\quad \text{and}\quad d = (d_1,\dots,d_T)^T, \tag{3.42}$$

then for homogeneous compound Poisson process, ML estimators of parameters obtained as

$$\begin{aligned}\hat{k} &= \frac{p_{-1}^T(p+d)}{p_{-1}^Tp_{-1}} - 1,\quad \hat{\sigma}_u^2 = \frac{e^Te}{T},\quad \hat{\lambda} = \frac{N(T)}{T},\\ \hat{\mu}_j &= \frac{Q^{(j)}(t_{j-1},t_j]}{N(t_{j-1},t_j]} = \frac{d_{t_j}-d_{t_{j-1}}}{N(t_{j-1},t_j]},\end{aligned}\tag{3.43}$$

and

$$\begin{aligned}\hat{\sigma}_j^2 &= \sum_{t\in\mathcal{I}^{(j)}}\frac{N(t-1,t]}{N(t_{j-1},t_j]}\left(\frac{Q^{(j)}(t-1,t]}{N(t-1,t]} - \frac{Q^{(j)}(t_{j-1},t_j]}{N(t_{j-1},t_j]}\right)^2\\ &= \frac{1}{N(t_{j-1},t_j]}\sum_{t\in\mathcal{I}^{(j)}}(d_t-d_{t-1}-\hat{\mu}_j)^2\end{aligned}\tag{3.44}$$

for $j = 1,\dots,r$, where $e = p + d - (1+\hat{k})p_{-1}$ is the unrestricted residual vector for the stock price. Instead of the traditional CAPM, we can estimate the discount rate by $\hat{k}$, using a very simple formula. In order to construct confidence intervals and test hypotheses for the parameters, we need to know the distribution of the estimators. In the following proposition, we list distributions which are related to the parameters.

Proposition 3.3. *The parameter estimators which are given by* (3.43) *and* (3.44) *of the model are consistent and for the estimators, the following probability laws* *hold*:

(1)

$$\frac{T\hat{\sigma}_u^2}{\sigma_u^2} \sim \chi^2(T-1), \tag{3.45}$$

(2)

$$\hat{\lambda}T \sim Pois(\lambda T), \tag{3.46}$$

(3)

$$\frac{N(t_{j-1},t_j]\hat{\sigma}_j^2}{\sigma_j^2} \,\Bigg|\, N(t_{j-1},t_j] \sim \chi^2(N(t_{j-1},t_j]-1), \quad j=1,\dots,r, \tag{3.47}$$

(4)

$$\frac{\mu_j-\hat{\mu}_j}{\hat{\sigma}_j} \,\Bigg|\, N(t_{j-1},t_j] \sim t(N(t_{j-1},t_j]-1), \quad j=1,\dots,r, \tag{3.48}$$

(5) *If* $H_0 : k = k_*$ *hypothesis is true, then*

$$LR(k_*) = T\ln\left(1+\frac{(\hat{k}-k_*)^2 p_{-1}^T p_{-1}}{e^T e}\right) \xrightarrow{d} \chi^2(1), \tag{3.49}$$

where by $Pois(\lambda)$, $\chi^2(n)$ *and* $t(n)$ *we denote a Poisson distribution with parameter* λ, *a chi-square distribution with* n *degrees of freedom, and a student* t *distribution with* n *degrees of freedom, respectively, and notation* LR *means the Likelihood Ratio statistic.*

If $k > 0$, then the time series of the price process P_t is explosive. Consequently, for a parameter estimator of the discount factor, we cannot apply the usual t-test directly. But, we can use (5) of the above Proposition 3 to test the hypothesis for the discount factor. From (1), (3) and (4) of the Proposition 3, one can easily obtain that $(1-\alpha)$ confidence intervals for corresponding parameters are given by

$$\begin{aligned} &\frac{T\hat{\sigma}_u^2}{\chi^2_{1-\alpha/2}(T-1)} \le \sigma_u^2 \le \frac{T\hat{\sigma}_u^2}{\chi^2_{\alpha/2}(T-1)}, \\ &\hat{\mu}_j - \hat{\sigma}_j t_{1-\alpha/2}(N(t_{j-1},t_j]-1) \\ &\qquad \le \mu_j \le \hat{\mu}_j + \hat{\sigma}_j t_{1-\alpha/2}(N(t_{j-1},t_j]-1) \,|\, N(t_{j-1},t_j], \end{aligned} \tag{3.50}$$

and

$$\frac{N(t_{j-1},t_j]\hat{\sigma}_j^2}{\chi^2_{1-\alpha/2}(N(t_{j-1},t_j]-1)} \le \sigma_j^2 \le \frac{N(t_{j-1},t_j]\hat{\sigma}_j^2}{\chi^2_{\alpha/2}(N(t_{j-1},t_j]-1)} \,\Bigg|\, N(t_{j-1},t_j] \tag{3.51}$$

for $j = 1,\dots,r$, where $\chi^2_\alpha(n)$ and $t_\alpha(n)$ are the α quantile of the $\chi^2(n)$ and $t(n)$ distributions, respectively. According to Johnson *et al.* (2005), an approximate

$(1-\alpha)$ confidence interval of parameter λ is given by

$$\hat{\lambda} + \frac{0.5}{T} z_{1-\alpha/2}^2 - z_{1-\alpha/2}\sqrt{\frac{1}{T}\hat{\lambda} + \frac{0.25}{T^2} z_{1-\alpha/2}^2}$$
$$\leq \lambda \leq \hat{\lambda} + \frac{0.5}{T} z_{1-\alpha/2}^2 + z_{1-\alpha/2}\sqrt{\frac{1}{T}\hat{\lambda} + \frac{0.25}{T^2} z_{1-\alpha/2}^2}. \tag{3.52}$$

It follows from Coles (2001) that an approximate $(1-\alpha)$ confidence interval for the discount factor k is given by

$$C_\alpha = \{k \in \mathbb{R} \,|\, \mathrm{LR}(k) \leq \chi_{1-\alpha}^2(1)\}. \tag{3.53}$$

For our model, this confidence interval is

$$\hat{k} - \sqrt{\frac{e^T e \left(\exp\left\{\frac{\chi_{1-\alpha}^2}{T}\right\} - 1\right)}{p_{-1}^T p_{-1}}}$$
$$\leq k \leq \hat{k} + \sqrt{\frac{e^T e \left(\exp\left\{\frac{\chi_{1-\alpha}^2}{T}\right\} - 1\right)}{p_{-1}^T p_{-1}}}. \tag{3.54}$$

To compare our model to Gordon's growth model and Yao's stochastic additive and geometric models, we need parameter estimations of the models, see Sec. 4. For this reason, let us consider Hurley & Johnson's (1998) stochastic additive and geometric models, which are generalization of the aforementioned models.

For Hurley & Johnson's (1998) stochastic geometric model, consecutive dividends are modeled by the following equation:

$$d_t = (1+g_i)d_{t-1} \quad \text{with probability } p_i, \ \ i = 1, \dots, n, \quad t = 1, \dots, T, \tag{3.55}$$

where g_i is a dividend growth rate at regime i, p_i is a probability, which indicates that the regime random variable I_t takes the value i, and d_t is the dividend at time t. To estimate parameters of the dividend growth rates $g_1, \dots, g_n$ and probabilities $p_1, \dots, p_n$, let us add random amounts, which follow a normal distribution with a mean of zero and a variance of σ_v^2, namely, $v_t \sim \mathcal{N}(0, \sigma_v^2)$, to Eq. (3.55). Then, the geometric model is determined by the following equations:

$$f(d_t \,|\, d_{t-1}, I_t = i) = \frac{1}{\sqrt{2\pi\sigma_v^2}} \exp\left\{-\frac{(d_t - (1+g_i)d_{t-1})^2}{2\sigma_v^2}\right\} \quad \text{and} \quad \mathbb{P}(I_t = i) = p_i \tag{3.56}$$

for $i = 1, \dots, n$ and $t = 1, \dots, T$, where f is a conditional density function of a dividend random variable at time t. Let us assume that the dividend error random variables $v_1, \dots, v_T$ and also the regime random variables $I_1, \dots, I_T$ are independent of one another. Let for $i = 1, \dots, n$, $z_{it} := 1_{\{I_t = i\}}$ be an indicator random variable of the regime random variable I_t, that is, if the regime random variable at time t is in regime i, then it takes value 1, otherwise it takes value 0, and $I := (I_1, \dots, I_T)^T$ be a regime random vector. Then, it follows that

$$\ln[\mathbb{P}(I = i)] = \sum_{t=1}^{T} \sum_{i=1}^{n} z_{it} \ln(p_i), \tag{3.57}$$

where $i = (i_1, \dots, i_T)$ is a realization of the regime random vector I. As a result, the log-likelihood function of a random vector $(d^T, I^T)^T$ is given by

$$\begin{aligned} \mathcal{L}(\theta) = &-\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln(\sigma_v^2) - \frac{1}{2\sigma_v^2}\sum_{t=1}^{T}\left(d_t - \sum_{i=1}^{n}(1+g_i)z_{it}d_{t-1}\right)^2 \\ &+ \sum_{t=1}^{T}\sum_{i=1}^{n} z_{it}\ln(p_i), \end{aligned} \tag{3.58}$$

where $\theta := (g_1, \dots, g_n, p_1, \dots, p_n, \sigma_v^2)^T$ is a parameter vector. From the log-likelihood function, one can obtain estimators of the parameters as

$$\hat{g}_i = \frac{\sum_{t=1}^{T} d_t d_{t-1} z_{it}}{\sum_{t=1}^{T} d_{t-1}^2 z_{it}}, \qquad \hat{p}_i = \frac{1}{T}\sum_{t=1}^{T} z_{it}, \quad \hat{\sigma}_v^2 = \frac{1}{T}\sum_{t=1}^{T}\left(d_t - \sum_{i=1}^{n}(1+\hat{g}_i)z_{it}d_{t-1}\right)^2 \tag{3.59}$$

for $i = 1, \dots, n$. In case of a dividend error random variable v_t and a price error random variable u_t that are correlated, Battulga (2022) obtains parameter estimators of the Gordon growth g rate and discount factor k.

For consecutive dividends of Hurley & Johnson's (1998) stochastic additive model, the following relation holds:

$$d_t = \Delta_i + d_{t-1} \quad \text{with probability } p_i, \quad i = 1, \dots, n, \;\; t = 1, \dots, T, \tag{3.60}$$

where Δ_i is a dividend increment at regime i, p_i is a probability, which indicates that the regime random variable I_t takes the value i, and d_t is the dividend at time t. In a similar manner as the geometric model, parameter estimators of the additive model can be obtained by

$$\begin{aligned} &\hat{\Delta}_i = \frac{1}{\sum_{t=1}^{T} z_{it}}\sum_{t=1}^{T}(d_t - d_{t-1})z_{it}, \quad \hat{p}_i = \frac{1}{T}\sum_{t=1}^{T} z_{it}, \\ &\hat{\sigma}_v^2 = \frac{1}{T}\sum_{t=1}^{T}\left(d_t - d_{t-1} - \sum_{i=1}^{n}\hat{\Delta}_i z_{it}\right)^2 \end{aligned} \tag{3.61}$$

for $i = 1, \dots, n$.

4. Data and Numerical Results

We start by applying the estimation method for the parameter estimation of our model, see Sec. 3.4. For means of illustration, we have chosen five companies from different sectors (Healthcare, Financial Services, Consumer, Technology), listed in the S&P 500 index. Our data covers a period from Q1 1980 to Q3 2021. All quarterly price and dividend data have been collected from Thomson Reuters Eikon and Yahoo Finance, respectively. In order to increase the number of price and dividend observation points, we take quarterly data instead of yearly data. That leads to $T = 166$ observations for Johnson & Johnson, PepsiCo, and Walmart. Because of the (limited) availability of dividend data for JPMorgan, its data starts from Q2 1984 ($T = 149$), whereas the observation of Apple starts from Q2 1987 ($T = 137$).

The dividend payments of the selected companies show different patterns. In particular, JPMorgan cut its dividend by a huge amount due to the 2008/2009 financial crises, where as Apple did not pay dividends over longer periods, whereas the other companies display continuously increasing dividend dynamics which are

Table 3. Parameter estimations at time Q2 2021 of our model.

Row	Parameters	J&J	JPMorgan	PepsiCo	Apple	Walmart
2	$\hat{\lambda}_i$	0.255	0.182	0.255	0.110	0.261
3	$\hat{k}_i$	2.41%	3.32%	2.23%	7.25%	2.27%
4	k_i^L	1.41%	1.17%	1.12%	4.86%	0.88%
5	k_i^U	3.42%	5.48%	3.34%	9.63%	3.65%
6	$(1+\hat{k}_i)^4 - 1$	10.01%	13.98%	9.23%	32.29%	9.38%
7	t_1	67	53	57	34	68
8	t_2	96	58	96	42	75
9	t_3	112	108	152	101	86
10	t_*	40	40	40	37	30
11	$m_{i,2} := \hat{\mu}_{i,2}/\hat{\mu}_{i,1}$	4.163	36.666	1.468	8.007	2.827
12	$m_{i,3} := \hat{\mu}_{i,3}/\hat{\mu}_{i,1}$	8.793	1.833	8.117	707.611	5.842
13	$m_{i,4} := \hat{\mu}_{i,4}/\hat{\mu}_{i,1}$	9.260	79.444	12.742	104.139	13.568
14	$m_{i,5} := \hat{\mu}_{i,5}/\hat{\mu}_{i,1}$	2.454	20.012	2.832	34.941	1.408
15	$s_{i,2} := \hat{\sigma}_{i,2}/\hat{\sigma}_{i,1}$	2.041	0.000	0.828	0.000	0.000
16	$s_{i,3} := \hat{\sigma}_{i,3}/\hat{\sigma}_{i,1}$	0.652	3.499	5.264	0.000	0.214
17	$s_{i,4} := \hat{\sigma}_{i,4}/\hat{\sigma}_{i,1}$	2.680	1.402	8.516	98.608	8.579
18	$s_{i,2} := \hat{\sigma}_{i,5}/\hat{\sigma}_{i,1}$	0.312	0.009	0.198	9.439	0.019
19	$\bar{\mu}_{i,1}$	0.046	0.047	0.048	0.014	0.025
20	$\bar{\sigma}_{i,1}$	0.008	0.138	0.029	0.004	0.019
21	AE_i	0.0005	0.1486	0.0004	1.1799	0.0008

Note: $\hat{\lambda}_i$ is the estimation of intensity; $\hat{k}_i$ is the estimation of the discount factor; $\hat{k}_i^L$ and $\hat{k}_i^U$ are the 95% the lower and upper bounds of the discount factor k_i; the 6th row corresponds to the yearly discount rate; t_j, $j = 1, 2, 3$ is the end date of the jth period; t_* is the period length which is used to model the fifth period dividends by linear regression; $m_{i,j}$ and $s_{i,j}$, $j = 2, \ldots, 5$ are the jth period multiplier of mean and standard deviation. They are used to model the mean and standard deviation of the new jth period; $\bar{\mu}_{i,1}$ and $\bar{\sigma}_{i,1}$ are the new 1st period mean and standard deviation and AE_i is the absolute error.

not affected by the 2008/2009 financial crises. For our model, we assume for all companies, that a default never occurs. For the selected companies, absolute errors AE_is (see Table 3, and its explanation below) are sufficiently small for a number of periods equal to 5 ($n = 4$). It should be noted that if there are some js, $j = 1, \ldots, n$ such that $t_j = t_{j+1}$, then the number of periods will decrease. t_0 corresponds to the first quarter, and t_4 corresponds to the last quarter of dividends of the selected firms. For example, if we aim to calculate Q1 2021 theoretical prices of the companies' stocks, then the last quarter corresponds to Q4 2020. Parameter estimations of the companies are based on Eqs. (3.43) and (3.44). To obtain the theoretical price at time Q3 2021, first we estimate the parameters of the zero-period pricing formula (4.1) that correspond to Q2 2021, then we calculate the next period's theoretical price at time Q3 2021 using the one-period pricing formula (4.2), and the parameter estimations.

In Table 3, we present results that correspond to Q2 2021. Since explanations are comparable for the other companies, we will give an explanation for Johnson & Johnson (J&J) only. For J&J, the first period covers quarters 0–66 ($t_0 = 0 < t \leq t_1 = 67$), the second period covers quarters 67–95 ($t_1 = 67 < t \leq t_2 = 96$), the third period covers quarters 97–111 ($t_2 = 97 < t \leq t_3 = 112$), the fourth period covers quarters 112–164 ($t_3 = 112 < t \leq t_4 = 165$), and the final period covers quarters 165–1000 ($t_4 = 165 < t \leq t_{\max} = 1000$), see rows 7–9th of Table 3. We give an explanation of $t_{\max} = 1000$ and choice of t_1, t_2 and t_3 below.

We model the random time of dividend changes of the five companies by a homogeneous Poisson process, and we estimate the parameters of the mean function $\mu_i(t) = \lambda_i t$, $i = 1, \ldots, 5$ by Eq. (3.43). Further, parameter estimates of the homogeneous Poisson process are provided in the second row of Table 3, from which we can deduce that, on average, the dividends of Johnson & Johnson change 25.5 times in 100 quarters.

For the parameter estimation period, we suppose that for $i = 1, \ldots, 5$ and $j = 1, \ldots, 5$, the dividend jump size random variables $Y_{i,1}^{(j)}, Y_{i,2}^{(j)}, \ldots$ follow identical normal distributions with means of $\mu_{i,j}$ and variances of $\sigma_{i,j}^2$ that is for $q = 1, 2, \ldots$, $Y_{i,q}^{(j)} \sim \mathcal{N}(\mu_{i,j}, \sigma_{i,j}^2)$.

The discount rates estimation at time Q2 2021 of the firms is presented in row 3 of Table 3, while the corresponding 95% confidence intervals are included in rows 4 and 5 below. To calculate confidence bands, we used Eq. (3.54). Since the discount rates express the average quarterly return of the companies, we need to convert them annual discount rates. The results are presented in row 6 of Table 3. Table 3 further illustrates average returns (2.41% for J&J) and return variability, as the return is supposed to lie within the (1.41%, 3.42%) interval with a 95% probability.

If we graphically analyze dividend payments of the companies (see Fig. 2), we can see that the last t_* dividend data is growing in a linear way. Therefore, for the fifth period in which we do not know the dividends of the companies, we model the dividends of the companies by two-variable linear regressions that are explained by

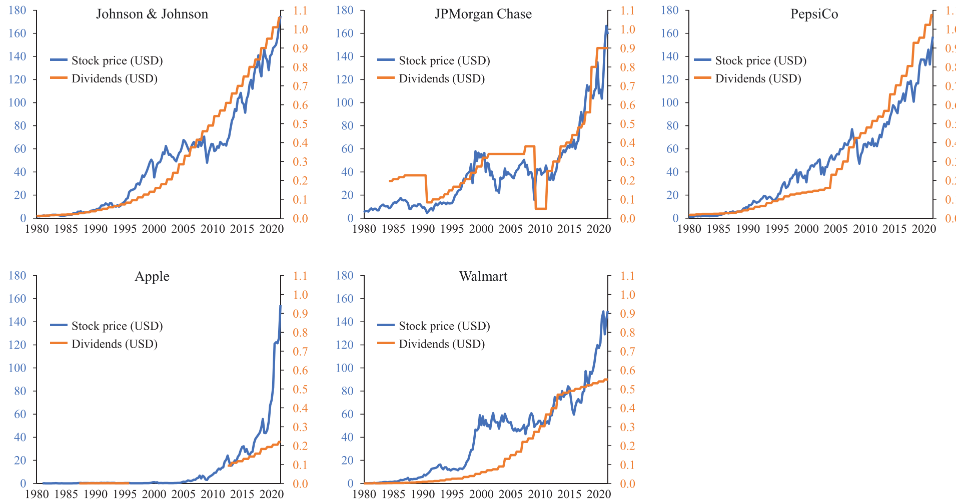

Fig. 2. Stocks and dividends of selected companies.

time and that are based on the last t_* dividends. For each firm, we provide values of t_* in row 10 of Table 3. Since we modeled the dividends of the companies by linear regression, for all sufficiently large m ($m > t_{\max}$) and $t \geq 0$, the terms $d_{t+m}/(1+\hat{k}_i)^m$ in the DDM pricing formula are almost zero. Therefore, it is reasonable to choose a value of $t_{\max}$ by 1000. That means we can omit dividends whose time exceeds $t + 1000$ from our model, because of $\sum_{m=1001}^{\infty} d_{t+m}/(1+\hat{k}_i)^m \approx 0$. Thus, instead of the fifth period, which covers the quarters t_4-infinity, we can approximate it by a period which covers quarters t_4-1000.

For each period, the parameter estimation of the mean and the standard deviation of the jump size random variables are calculated by Eqs. (3.43) and (3.44). In rows 11–14th of Table 3, we provide ratios of mean dividend jumps that correspond to the dividend jumps of the second to the fifth period to the first period mean dividend jump. Let us denote them by $m_{i,j}$, $i = 1, \ldots, 5$ and $j = 2, \ldots, 5$ and call them ith company's jth period multipliers of mean. Also, in rows 15–18th of the same table, we present ratios of standard deviations that correspond to the dividend jumps of the second to the fifth period of dividend jumps to the first period standard deviation of dividend jump. We denote them by $s_{i,j}$, $i = 1, \ldots, 5$ and $j = 2, \ldots, 5$ and call them ith company's jth period multipliers of standard deviation.

Now we assume that the calculation time frame starts at time $\bar{t}_0 := t_4 - t_1$. In this case, the new first period covers the quarters from $\bar{t}_0 + 1$ to $\bar{t}_1$ ($\bar{t}_0 < t \leq \bar{t}_1 := t_4$), the new second period covers the quarters from $\bar{t}_1 + 1$ to $\bar{t}_2$ ($\bar{t}_1 < t \leq \bar{t}_2 := t_4 - t_1 + t_2$), the new third period covers the quarters $\bar{t}_2 + 1$ to $\bar{t}_3$ ($\bar{t}_2 < t \leq \bar{t}_3 := t_4 - t_1 + t_3$), the new fourth period covers the quarters $\bar{t}_3 + 1$ to $\bar{t}_4$ ($\bar{t}_3 < t \leq \bar{t}_4 := 2t_4 - t_1$) and the new final period covers the quarters $\bar{t}_4 + 1$ to 1000 ($\bar{t}_4 < t \leq 1000$). In order to obtain future means and standard deviations of dividend jump size random variables for each period and each company, first, we calculate the new first period means and standard deviations of random dividend jumps of the companies using the last t_1+1 dividends (from $\bar{t}_0$ to $\bar{t}_1$), and Eqs. (3.43) and (3.44).

Let us denote the means and standard deviations by $\bar{\mu}_{i,1}$ and $\bar{\sigma}_{i,1}$, $i = 1,\ldots,5$. We assume that for the new jth period ($j = 2,\ldots,5$), the means and standard deviations of the dividend jump size random variables equal means and standard deviations of the random dividend jumps of the new first period times the multipliers of the mean and standard deviation, i.e. $\bar{\mu}_{i,j} = \bar{\mu}_{i,1}m_{i,j}$ and $\bar{\sigma}_{i,j} = \bar{\sigma}_{i,1}s_{i,j}$, $i = 1,\ldots,5$ and $j = 2,\ldots,5$. That means period j multipliers are the same for old and new periods that start at time $t_0 = 0$ and $\bar{t}_0 = t_4 - t_1$, respectively.

Under this assumption, the new dividend jump size random variables of the companies follow normal distributions, i.e. $Y_{i,1}^{(j)} \sim \mathcal{N}(\bar{\mu}_{i,j}, \bar{\sigma}_{i,j}^2)$. For example, the new second period dividend jump size random variables of Johnson & Johnson follow a normal distribution with a mean of $\bar{\mu}_{1,2} = 0.046 \times 4.163 = 0.190$ and a standard deviation of $\bar{\sigma}_{1,2} = 0.008 \times 2.041 = 0.017$, that is, for $q = 1,2,\ldots$, $Y_{1,q}^{(2)} \sim \mathcal{N}(0.190, 0.017^2)$.

To calculate the zero-period theoretical price of the companies' stock, let us reconsider the theoretical price formula (3.37) of a company. It follows from equation (3.37) that one can obtain the following equation, which corresponds to the selected firms:

$$\mathbb{E}(P_{i,0}) \approx \hat{\lambda}_i F_0(1) \left\{ \frac{d_{i,0}}{\hat{\lambda}_i} + \bar{\mu}_{i,1}F_0(0,t_1] + \bar{\mu}_{i,2}F_0(t_1,t_2] \right. \\ \left. + \bar{\mu}_{i,3}F_0(t_2,t_3] + \bar{\mu}_{i,4}F_0(t_3,t_4] + \bar{\mu}_{i,5}F_0(t_4,t_{\max}] \right\} \tag{4.1}$$

for $i = 1,\ldots,5$. For fixed t_4 and $t_{\max}$, the values of t_1, t_2 and t_3 are obtained by minimizing the absolute error of the real and the zero-period theoretical price at time Q2 2021, where the theoretical price of the ith, $i = 1,\ldots,5$ company is calculated by Eq. (4.1). The absolute error is defined by $\mathrm{AE}_i = |P_i^{Q2/21} - \mathbb{E}(P_{i,0})|$, where $P_i^{Q2/21}$ is the real and $\mathbb{E}(P_{i,0})$ is the zero-period theoretical price at time Q2 2021. In row 21 of Table 3, we present the absolute errors for each company.

After obtaining the parameters of the model, the one-period theoretical prices at Q2 2021 have to be calculated. It follows from Eq. (3.37) that for the selected firms, the one-period theoretical price that corresponds to Q3 2021 is given by the following equation:

$$\mathbb{E}(P_{i,1}) \approx \hat{\lambda}_i F_1(2) \left\{ \frac{d_{i,0}}{\hat{\lambda}_i} + \bar{\mu}_{i,1}(1 + F_1(1,t_1]) + \bar{\mu}_{i,2}F_1(t_1,t_2] \right. \\ \left. + \bar{\mu}_{i,3}F_1(t_2,t_3] + \bar{\mu}_{i,4}F_1(t_3,t_4] + \bar{\mu}_{i,5}F_1(t_4,t_{\max}] \right\}. \tag{4.2}$$

The second row of Table 4 shows the one-period theoretical price of the companies based on our model. In the 3rd row, we provide the real price of the companies as of Q3 2021. In rows 4–6, we provide the results of the Monte Carlo simulation (of 5000

Table 4. Results at time Q3 2021 of our model.

Row	Parameters	J&J	JPMorgan	PepsiCo	Apple	Walmart
2	$\mathbb{E}(P_{i,1})$	168.868	170.604	150.045	136.070	144.497
3	$P_i^{\mathrm{Q3/21}}$	175.040	159.490	157.090	154.300	149.250
4	$\overline{P}_{i,1}$	167.894	170.181	148.834	135.658	144.003
5	$\overline{P}_{i,1}^L$	121.098	67.143	108.487	28.341	93.289
6	$\overline{P}_{i,1}^U$	220.892	305.843	193.866	277.653	203.452
7	σ_i	27.123	59.617	22.090	76.593	29.231
8	$b_{i,3}$	0.371	0.540	0.408	0.751	0.365
9	$\kappa_{i,4}$	0.311	0.401	0.394	0.634	0.212

Note: $\mathbb{E}(P_{i,1})$ is the theoretical stock price at time Q3 2021; $P_i^{\mathrm{Q3/21}}$ is the real price at time Q3 2021; $\overline{P}_{i,1}$ is the average price that corresponds to the Monte Carlo simulation; $\overline{P}_{i,1}^L$ and $\overline{P}_{i,1}^U$ are the Monte Carlo 95% lower and upper bounds of the price and σ_i, $b_{i,3}$ and $\kappa_{i,4}$ are the standard deviation, skewness, and excess kurtosis.

runs). In the 4th row, we show the average of the simulation, which approximates the 2nd row by the law of large numbers. Rows 5 and 6 correspond to the 95% confidence level of the Monte Carlo lower and upper bound of the random price of the companies, respectively. From the lower and upper bounds, we can see that the real prices of all companies are inside the confidence intervals. In rows 7–9 of Table 4, we provide the results of the standard deviation, skewness, and kurtosis of the one-period price of the companies, respectively.

Note that for JPMorgan and Apple, standard deviations and confidence bands are large as compared to other companies, because of huge jumps in their dividends. From the skewness and kurtosis, we can deduce that the shape of the distribution of the companies is positively skewed and fat-tailed compared to the normal distribution. Results of a comparative analysis of our model and the other models for Q3-Q4 2020 and Q1-Q2 2021 are provided in Table 6.

In order to compare our model to other DDMs which are used in practice, let us briefly review the other models. In the Gordon growth model, it is assumed that future dividends grow at a constant rate, say g. In this case, the price at time t of a dividend paying company is given by the following equation:

$$P_t = \frac{(1+g)^{t+1}}{k-g} d_t, \quad t = 0, 1, \ldots, \tag{4.3}$$

where g is the growth rate of dividends, d_t is the dividend at time t, and k is the discount rate of the company.

Hurley & Johnson (1994) introduced a new family of valuation models. They assume that either the firm will increase dividends by a constant amount or keep dividends the same. Since dividends take only two values with specific probabilities, their model is referred to as a binomial stochastic DDM. Thus, the binomial stochastic DDM does not take into account the possibility of decreasing dividends, what can be considered a serious shortcoming of the model, because there is sufficient empirical proof that dividend decreases can occur.

Within the Hurley–Johnson stochastic DDM framework, Yao (1997) derived a model that allows for a decrease in dividends. Yao's model is called a trinomial stochastic DDM because it allows for an increase in dividends, no change in dividends, and a decrease in dividends. There are two versions of the trinomial stochastic DDM, which are briefly distinguished below.

The first version is the trinomial additive stochastic model. For this model, the firm's dividend stream is given by the following equation:

$$d_{t+1} = \begin{cases} d_t + \Delta_u, & \text{with probabilty } p_u, \\ d_t + \Delta_d, & \text{with probabilty } p_d, \\ d_t, & \text{with probabilty } 1 - p_u - p_d \end{cases} \qquad t = 0, 1, \ldots, \tag{4.4}$$

where d_t is the dividend at time t, Δ_u is the dividend increment in the upward scenario, Δ_d is the dividend increment in the downward scenario, p_u is the probability of the dividend increase (upward scenario), and p_d is the probability of the dividend decrease (downward scenario). Under the dividend stream, Yao (1997) obtains a theoretical price formula at time t which is given by the following equation:

$$P_t = \frac{1}{k}\left\{d_0 + \left(t + \frac{1+k}{k}\right)(\Delta_u p_u + \Delta_d p_d)\right\}, \quad t = 0, 1, \ldots. \tag{4.5}$$

The second version is a trinomial geometric stochastic model. For the second model, its dividend stream is given by the following equation:

$$d_{t+1} = \begin{cases} (1 + g_u)d_t, & \text{with probabilty } p_u, \\ (1 + g_d)d_t, & \text{with probabilty } p_d, \\ d_t & \text{with probabilty } 1 - p_u - p_d \end{cases} \qquad t = 0, 1, \ldots, \tag{4.6}$$

where d_t is the dividend at time t, g_u is the dividend growth rate in an upward scenario, g_d is the dividend growth rate in a downward scenario, p_u is the probability of a dividend increase (upward scenario), and p_d is the probability of a dividend decrease (downward scenario). In this dividend stream, Yao (1997) shows that the theoretical price at time t of a stock is

$$P_t = \frac{(1 + g_u p_u + g_d p_d)^{t+1}}{k - g_u p_u - g_d p_d} d_t, \quad t = 0, 1, \ldots. \tag{4.7}$$

It should be noted that if $p_d = 0$, then Yao's trinomial model becomes Hurley and Johnson's binomial model, and if $p_u = 1$, then Yao's trinomial geometric model becomes Gordon's growth model.

In row 2 of Table 5, we provide the companies' dividends at time T. The 3rd row of Table 5 provides the number of quarters included in the analysis. Dividends of Johnson & Johnson, PepsiCo and Walmart start at time Q1 1980, whereas the dividends of JPMorgan and Apple start at time Q2 1984 and Q2 1986, respectively.

Table 5. Theoretical price at time Q3 2021 of Gordon model and Yao's models.

Row	Parameters	J&J	JPMorgan	PepsiCo	Apple	Walmart
2	d_{T_i}	1.060	0.900	1.075	0.220	0.550
3	T_i	165	148	165	136	165
4	k_i	2.55%	2.86%	2.40%	9.43%	2.41%
5	g_i	1.68%	1.26%	1.88%	2.14%	1.24%
6	$p_{i,u}$	0.255	0.169	0.248	0.103	0.248
7	$p_{i,d}$	0.000	0.014	0.006	0.007	0.012
8	$\Delta_{i,u}$	0.025	0.047	0.026	0.016	0.013
9	$\Delta_{i,d}$	0.000	0.237	0.001	0.001	0.002
10	$g_{i,u}$	6.97%	15.06%	7.84%	8.98%	5.24%
11	$g_{i,d}$	0.00%	80.65%	0.19%	100.00%	77.36%
12	$\mathbb{E}^G(P_{i,0})$	123.594	57.078	211.907	3.082	47.616
13	$\mathbb{E}^Y_a(P_{i,0})$	51.500	37.420	56.261	2.531	28.702
14	$\mathbb{E}^Y_g(P_{i,0})$	138.407	64.878	243.549	2.384	26.984

Note: d_{T_i} is the dividend at time Q3 2021; T_i is the number of quarters; k_i is the quarterly discount factor; g_i is the quarterly dividend growth rate; $p_{i,u}$ is the probability of dividend increase; $p_{i,d}$ is the probability of dividend decrease; Δ_u is the dividend increment in an upward scenario; Δ_d is the dividend increment in a downward scenario; $g_{i,u}$ is the quarterly dividend growth rate in an upward scenario; $g_{i,d}$ is quarterly dividend growth rate in a downward scenario; $\mathbb{E}^G(P_{i,0})$ is the theoretical price of the Gordon model; $\mathbb{E}^Y_a(P_{i,0})$ is the theoretical price of Yao's additive model and $\mathbb{E}^Y_g(P_{i,0})$ is the theoretical price of Yao's additive model.

Dividends of all companies end by Q3 2021. The 4th row of the table shows discount rates at that time (see also the discount rate at time Q2 2021 in the 3rd row of Table 3). In the 5th row of the table, we provide dividend growth rates of the companies. The growth rates are estimated by Eq. (3.59), which corresponds to only one scenario. In rows 6 and 7 of Table 5, we present the probabilities of a dividend increase p_u and a dividend decrease p_d, both obtained by using Eq. (3.59), which correspond to upward/downward/constant scenario. For example, JPMorgan's dividends increased 25 times, remained constant 121 times and decreased only two times. So, $p_{2,u} = 25/148 = 0.169$ and $p_{2,d} = 2/149 = 0.014$. We estimate Δ_u (Δ_d) by calculating changes in dividends for each quarter of dividend increase (decrease), and then average the dividend changes, see Eq. (3.61). The averages for the companies are provided in rows 8 and 9 of Table 4 above. Yao's growth rates, which correspond to the upward and downward scenarios, are estimated by Eq. (3.59). We provide Yao's growth rates for each company in rows 10 and 11 of Table 5. Finally, rows 12–15 correspond to the theoretical values of the deterministic Gordon growth model, Yao's stochastic additive model, and Yao's stochastic geometric model, respectively.

It should be noted that because of the particular discount rate of Apple, its theoretical value is very low compared to its real price, which is given in the 4th row of Table 4. In order to compare the theoretical prices of stocks resulting from the models to the real prices, we combine the information in Table6. According to the DDM approach, the growth rate expectation of dividends must be smaller than the discount rate. If this condition does not hold, we substitute "–" in Table 6. In the case of the Gordon model, it arises when the discount factor k is lower than the growth rate g, see the theoretical prices of JPMorgan in Table 6. In case of the trinomial geometric model, the condition collapses because the growth rate expectation $g_u p_u + g_d p_d$ exceeds the discount factor k, see the theoretical prices of JPMorgan and PepsiCo in Table 6.

The theoretical price at time Q3 2021 using our model and the real price at time Q3 2021 are given in rows 2–3 in Table 4. From Table 4– 6, we can deduce that our model dominates the other models. A look into the graphical illustration (see Fig. 3) emphasizes the dominance of our model compared to the other models. Note that the other models are particularly sensitive to their input parameters, see Tables 4–6. Because theoretical price at time $t-1$ of our model is rather close to

Table 6. Comparison of the models at times Q3 2020, Q4 2020, Q1 2021, and Q2 2021.

Row	Models	J&J	JPMorgan	PepsiCo	Apple	Walmart
2	$P_i^{\mathrm{Q3/20}}$	148.590	103.520	138.760	120.960	142.830
3	$\mathbb{E}(P_{i,1})$	149.654	113.004	158.319	88.472	123.481
4	$\mathbb{E}^G(P_{i,0})$	240.894	—	526.966	2.122	41.153
5	$\mathbb{E}_a^Y(P_{i,0})$	59.684	88.865	60.033	1.811	25.153
6	$\mathbb{E}_g^Y(P_{i,0})$	315.244	—	856.634	1.719	24.444
7	$P_i^{\mathrm{Q4/20}}$	150.270	122.340	145.850	122.250	148.910
8	$\mathbb{E}(P_{i,1})$	153.574	103.094	145.071	131.098	146.078
9	$\mathbb{E}^G(P_{i,0})$	215.867	179.391	257.977	2.747	37.368
10	$\mathbb{E}_a^Y(P_{i,0})$	60.023	59.247	56.035	2.285	24.077
11	$\mathbb{E}_g^Y(P_{i,0})$	316.295	197.146	401.180	2.147	23.389
12	$P_i^{\mathrm{Q1/21}}$	156.100	150.910	133.030	121.420	129.120
13	$\mathbb{E}(P_{i,1})$	155.251	124.079	148.180	135.483	152.464
14	$\mathbb{E}^G(P_{i,0})$	167.331	61.720	1062.293	3.569	85.522
15	$\mathbb{E}_a^Y(P_{i,0})$	57.355	37.619	69.668	2.876	36.987
16	$\mathbb{E}_g^Y(P_{i,0})$	246.859	66.189	—	2.670	34.661
17	$P_i^{\mathrm{Q2/21}}$	165.970	166.440	147.840	125.890	141.850
18	$\mathbb{E}(P_{i,1})$	158.603	154.965	172.725	131.607	131.079
19	$\mathbb{E}^G(P_{i,0})$	160.793	45.434	403.004	4.519	56.056
20	$\mathbb{E}_a^Y(P_{i,0})$	55.164	31.540	61.456	3.368	30.921
21	$\mathbb{E}_g^Y(P_{i,0})$	171.741	49.073	404.539	3.124	29.032

Note: P_i^t is the real price at time t; $\mathbb{E}(P_{i,1})$ is the theoretical price of our model; $\mathbb{E}^G(P_{i,0})$ is the theoretical price of the Gordon model; $\mathbb{E}_a^Y(P_{i,0})$ is the theoretical price of Yao's additive model and $\mathbb{E}_g^Y(P_{i,0})$ is the theoretical price of Yao's additive model. The 2–21st rows correspond to prices at times Q3 2020, Q4 2020, Q1 2021 and Q2 2021, respectively.

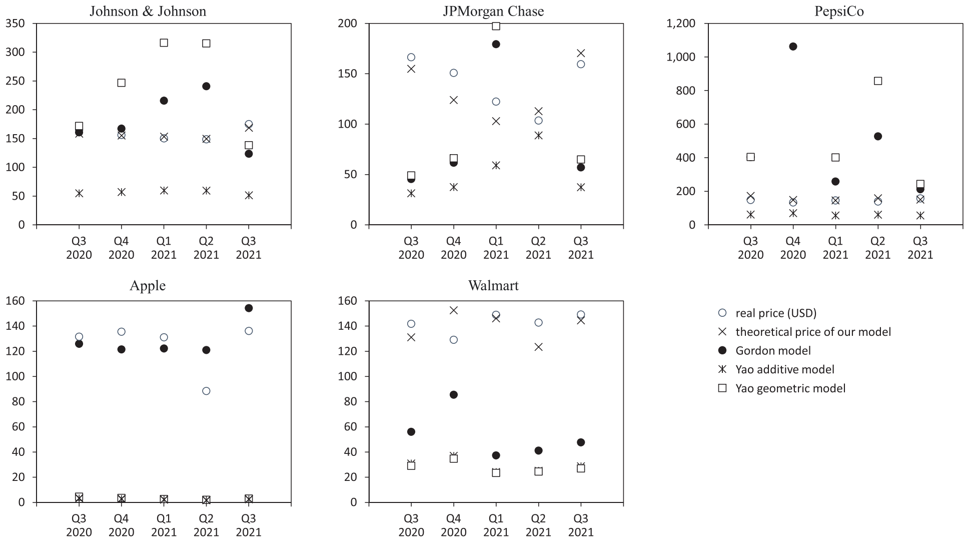

Fig. 3. Comparison of the models.

its real price (since we use absolute error terms, see row 21 in Table 3), the next period's theoretical price at time t of our model will be close to the real price as compared to other models. This is another reason why our model dominates the other models.

5. Conclusion

In this paper, we have introduced a stochastic multi-period DDM that includes (i) a compound nonhomogeneous Poisson process for dividend growth and (ii) the probability of firm default. We present expressions for the conditional moments (a firm that defaults) and unconditional moments (a firm that never defaults) of the multi-period price of a firm's stock — assuming that dividends evolve stochastically according to a compound nonhomogeneous Poisson process over each period. Moreover, we obtained ML estimators of our model's parameters, and constructed formulas of confidence intervals of the parameters.

The stochastic $(n+1)$-period model introduced in this paper has various advantages for analysts. One benefit of this model is its flexibility: the calculations can be automated in a valuation tools analysts use for their forecasts. As shown, our model is relatively close to real price data, and it dominates the practical models, namely Gordon's model and Yao's models. Overall, the model can be considered an enhanced tool for the calculation of (moments of) future stock prices of firms with different growth prospects in the long run.

For future research, it will be interesting to consider cases, where the stock price P_t and the dividend increment $Q(t-1,t]$ are dependent. Consequently, future work should focus on developing models in which the random error of the stock price u_t, and the dividend jump random variable Y_i are correlated.

Technical Annex

Here we give proofs of the propositions.

Proof of Proposition 1. First, we assume that the default occurs at ith period (where $j \leq i$, that is, $t_{i-1} < \tau \leq t_i$). Then, the accumulated dividends until the default is given by

$$
\begin{aligned}
&m = 1 : d_{t+1} = d_t + Q^{(j)}(t, t+1],\\
&m = 2 : d_{t+2} = d_t + Q^{(j)}(t, t+2],\\
&\qquad \vdots\\
&m = t_j - t : d_{t_j} = d_t + Q^{(j)}(t, t_j],\\
&m = t_j - t + 1 : d_{t_j+1} = d_t + Q^{(j)}(t, t_j] + Q^{(j+1)}(t_j, t_j + 1],\\
&m = t_j - t + 2 : d_{t_j+2} = d_t + Q^{(j)}(t, t_j] + Q^{(j+1)}(t_j, t_j + 2],\\
&\qquad \vdots\\
&m = t_{j+1} - t : d_{t_{j+1}} = d_t + Q^{(j)}(t, t_j] + Q^{(j+1)}(t_j, t_{j+1}],\\
&\qquad \vdots\\
&m = t_{i-1} \vee t - t + 1 : d_{t_{i-1}+1}\\
&\quad = d_t + Q^{(j)}(t, t_j] + Q^{(j+1)}(t_j, t_{j+1}] + \cdots + Q^{(i)}(t_{i-1} \vee t, t_{i-1} \vee t + 1],\\
&m = t_{i-1} \vee t - t + 2 : d_{t_{i-1}+2}\\
&\quad = d_t + Q^{(j)}(t, t_j] + Q^{(j+1)}(t_j, t_{j+1}] + \cdots + Q^{(i)}(t_{i-1} \vee t, t_{i-1} \vee t + 2],\\
&\qquad \vdots\\
&m = \tau - t - 1 : d_{\tau-1}\\
&\quad = d_t + Q^{(j)}(t, t_j] + Q^{(j+1)}(t_j, t_{j+1}] + \cdots + Q^{(i)}(t_{i-1} \vee t, \tau - 1],
\end{aligned}
$$

where $a \vee b = \max(a, b)$. Therefore, the random price of a common stock at time t (see Williams (1938)) is given by

$$
P_t = \sum_{m=1}^{\tau-t-1} \frac{d_{t+m}}{(1+k)^m} = \sum_{m=1}^{\tau-t-1} \frac{d_t}{(1+k)^m} + \sum_{m=1}^{t_j-t} \frac{Q^{(j)}(t, t+m]}{(1+k)^m}
$$

$$+\sum_{m=t_j-t+1}^{\tau-t-1}\frac{Q^{(j)}(t,t_j]}{(1+k)^m}+\sum_{m=t_j-t+1}^{t_{j+1}-t}\frac{Q^{(j+1)}(t_j,t+m]}{(1+k)^m}$$
$$+\sum_{m=t_{j+1}-t+1}^{\tau-t-1}\frac{Q^{(j+1)}(t_j,t_{j+1}]}{(1+k)^m}+\cdots+\sum_{m=t_{i-1}\vee t-t+1}^{\tau-t-1}\frac{Q^{(i)}(t_{i-1}\vee t,t+m]}{(1+k)^m}. \tag{5.1}$$

The first term of the above equation can be expressed as

$$\sum_{m=1}^{\tau-t-1}\frac{d_t}{(1+k)^m}=\frac{1}{k}\left(1-\frac{1}{(1+k)^{\tau-t-1}}\right)d_t=F_t(\tau,t+1]d_t.$$

Taking the second and third terms of the right-hand side of Eq. (5.1), we have

$$\sum_{m=1}^{t_j-t}\frac{Q^{(j)}(t,t+m]}{(1+k)^m}+\sum_{m=t_j-t+1}^{\tau-t-1}\frac{Q^{(j)}(t,t_j]}{(1+k)^m}=\sum_{m=t+1}^{t_j}F_t(\tau,m]Q^{(j)}(m-1,m]$$

and for $u=j+1,\dots,i-1$, one has

$$\sum_{m=t_{u-1}-t+1}^{t_u-t}\frac{Q^{(u)}(t_{u-1},t+m]}{(1+k)^m}+\sum_{m=t_u-t+1}^{\tau-t-1}\frac{Q^{(u)}(t_{u-1},t_u]}{(1+k)^m}$$
$$=\sum_{m=t_{u-1}+1}^{t_u}F_t(\tau,m]Q^{(u)}(m-1,m].$$

Finally, the last term of the right-hand side of Eq. (5.1) becomes

$$\sum_{m=t_{i-1}\vee t-t+1}^{\tau-t-1}\frac{Q^{(i)}(t_{i-1}\vee t,t+m]}{(1+k)^m}=\sum_{m=t_{i-1}\vee t+1}^{\tau-1}F_t(\tau,m]Q^{(i)}(m-1,m].$$

Summarizing the above calculations, we obtain the proposition. □

Proof of Proposition 2. On $A=\{\tau=s\}$ and $B=\{N(t_1)=z_{t_1},\dots,N(t_{j-1})=z_{t_{j-1}},N(t)=z_t,N(t+1)=z_{t+1},\dots,N(s-1)=z_{s-1}\}$ events, the stock price at time t becomes

$$P_t=R_s[z_{t_1},z_{t_2},\dots,z_{t_{j-1}},z_t]$$
$$+Q_s^{(j)}(z_t:z_{t_j}]+Q_s^{(j+1)}(z_{t_j}:z_{t_{j+1}}]+\cdots+Q_s^{(i)}(z_{t_{i-1}\vee t}:z_{s-1}].$$

Since the random default time τ, the Poisson process $N(t)$, and the jump size $Y_i^{(j)}$ are independent, we have

$$\mathbb{P}(P_t\le x\,|\,A\cap B)$$
$$=\mathbb{P}(R_s[z_{t_1},z_{t_2},\dots,z_{t_{j-1}},z_t]$$
$$+Q_s^{(j)}(z_t:z_{t_j}]+Q_s^{(j+1)}(z_{t_j}:z_{t_{j+1}}]+\cdots+Q_s^{(i)}(z_{t_{i-1}\vee t}:z_{s-1}]\le x).$$

By the independent increment property of a nonhomogeneous Poisson process, for any integers $z_1 \leq z_2 \leq \cdots \leq z_n$, it can be shown that the following relation holds:

$$\begin{aligned}&\mathbb{P}(N(t_1) = z_1, N(t_2) = z_2, \ldots, N(t_n) = z_n)\\&\qquad = e^{-\mu(t_n)} \frac{\mu(t_1)^{z_1}}{z_1!} \frac{\mu(t_1, t_2]^{z_2 - z_1}}{(z_2 - z_1)!} \cdots \frac{\mu(t_{n-1}, t_n]^{z_n - z_{n-1}}}{(z_n - z_{n-1})!}.\end{aligned}$$

Therefore, we obtain the first part of the proposition. Let $s > t_n$. We define

$$\begin{aligned}P_t^{(s)} &= F_t(t+1)d_t + \sum_{m=t+1}^{t_j} F_t(m)Q^{(j)}(m-1, m]\\&\quad + \sum_{m=t_j+1}^{t_{j+1}} F_t(m)Q^{(j+1)}(m-1, m] + \cdots + \sum_{m=t_n}^{s} F_t(m)Q^{(n+1)}(m-1, m].\end{aligned}$$

It can be shown that a characteristic function of a dividend increment from time t_{j-1} to time t_j corresponding to the jth period is given by

$$\phi_{Q^{(j)}(t_{j-1}, t_j]}(u) = \exp\{\mu(t_{j-1}, t_j](\phi_{Y_1^{(j)}}(u) - 1)\},$$

where $\phi_{Y_1^{(j)}}(u) = \mathbb{E}(\exp\{iuY_1^{(j)}\})$ is a characteristic function of a random variable $Y_1^{(j)}$. Therefore, the characteristic function of the random variable $P_t^{(s)}$ is obtained as

$$\phi_{P_t^{(s)}}(u) := \mathbb{E}(e^{iuP_t^{(s)}}) = \exp\{h_t^*(u \mid s)\},$$

where

$$\begin{aligned}h_t^*(u \mid s) &:= iuF_t(t+1)d_0 + \mu(t_1)(\phi_{Y_1^{(1)}}(uF_t(t+1)) - 1) + \cdots\\&\quad + \mu(t_{j-2}, t_{j-1}](\phi_{Y_1^{(j-1)}}(uF_t(t+1)) - 1)\\&\quad + \mu(t_{j-1}, t](\phi_{Y_1^{(j)}}(uF_t(t+1)) - 1)\\&\quad + \sum_{m=t+1}^{t_j} \mu(m-1, m](\phi_{Y_1^{(j)}}(uF_t(m)) - 1)\\&\quad + \sum_{m=t_j+1}^{t_{j+1}} \mu(m-1, m](\phi_{Y_1^{(j+1)}}(uF_t(m)) - 1)\\&\quad + \cdots + \sum_{m=t_n}^{s} \mu(m-1, m](\phi_{Y_1^{(n+1)}}(uF_t(m)) - 1).\end{aligned}$$

For the distribution function of the random variable $P_t^{(s)}$, similarly to the first part of the Proposition, we can get

$$\mathbb{P}(P_t^{(s)} \leq x) = \sum_{z_{t_1}=0,\dots,z_{t_j}=z_{t_{j-1}},z_{t+1}=z_t,\dots,z_s=z_{s-1}} \Pr(R_\infty[z_{t_1}, z_{t_2}, \dots, z_{t_{j-1}}, z_t]$$
$$+ Q_\infty^{(j)}(z_t : z_{t_j}] + Q_\infty^{(j+1)}(z_{t_j} : z_{t_{j+1}}] + \cdots + Q_\infty^{(n+1)}(z_{t_n} : z_s] \leq x)e^{-\mu(s)}$$
$$\times \frac{\mu(t_1)^{z_{t_1}}}{z_{t_1}!} \frac{\mu(t_1, t_2]^{z_{t_2}-z_{t_1}}}{(z_{t_2} - z_{t_1})!} \cdots \frac{\mu(t_{j-1}, t]^{z_t - z_{t_{j-1}}}}{(z_t - z_{t_{j-1}})!}$$
$$\times \prod_{m=t+1}^{s} \frac{\mu(m-1, m]^{z_m - z_{m-1}}}{(z_m - z_{m-1})!}.$$

Since $\lim_{s\to\infty} \phi_{P_t^{(s)}} = \phi_{P_t}$, according to the continuity Theorem (see Billingsley (1995)), we have

$$F_{P_t}(x) = \mathbb{P}(P_t \leq x) = \lim_{s\to\infty} \mathbb{P}(P_t^{(s)} \leq x). \qquad \square$$

Proof of Proposition 3. For the parameter estimator of the discount factor, which is given by Eq. (3.43), the estimator is consistent:

$$\operatorname*{plim}_{T\to\infty}\{\hat{k}\} = \operatorname*{plim}_{T\to\infty}\left\{k + \frac{p_{-1}^T u}{p_{-1}^T p_{-1}}\right\} = k,$$

where plim is the probability limit. Using Chebyshev's inequality, we obtain that the rest of the estimators are consistent. The unrestricted residual can be expressed by

$$e = \mathrm{Mu},$$

where $M := I_T - \frac{p_{-1}p_{-1}^T}{p_{-1}^T p_{-1}}$ is an idempotent and symmetric matrix, and its rank is $\mathrm{rank}(M) = \mathrm{tr}(M) = T-1$. Thus, for the unrestricted residual sum of square, we have

$$\frac{e^T e}{\sigma_u^2} = \frac{u^T \mathrm{Mu}}{\sigma_u^2} \sim \chi^2(T-1).$$

As a result, (1) holds. Since $N(T)$ is Poisson distributed with λT parameter, (2) is true. Now we fix j, $j = 1, \dots, r$. From Eq. (3.44), it can be shown that

$$\hat{\sigma}_j^2 = \frac{1}{N(t_{j-1}, t_j]}\left(\sum_{t\in\mathcal{I}^{(j)}} \frac{(Q^{(j)}(t-1,t])^2}{N(t-1,t]} - \frac{(Q^{(j)}(t_{j-1},t_j])^2}{N(t_{j-1},t_j]}\right).$$

If we define

$$Z^{(j)}(t-1,t] := \frac{Q^{(j)}(t-1,t]}{\sqrt{N(t-1,t]}}$$
$$\stackrel{d}{=} \frac{1}{\sqrt{N(t-1,t]}} \sum_{i=1}^{N(t-1,t]} Y_i^{(j)}, \quad \text{for } t = t_{j-1}+1, \dots, t_j,$$

then $Z^{(j)}(t-1,t]$ are independent and

$$Z^{(j)}(t-1,t]\,|\,N(t-1,t] \sim \mathcal{N}(\sqrt{N(t-1,t]}\mu_j, \sigma_j^2).$$

As $Q^{(j)}(t_{j-1},t_j] = \sum_{t\in\mathcal{I}^{(j)}} \sqrt{N(t-1,t]}Z^{(j)}(t-1,t]$, the estimator of parameter σ_j^2 can be written by

$$\hat{\sigma}_j^2 = \frac{1}{N(t_{j-1},t_j]}\sum_{t\in\mathcal{I}^{(j)}}(Z^{(j)}(t-1,t])^2 - \frac{1}{N^2(t_{j-1},t_j]}\left(\sum_{t\in\mathcal{I}^{(j)}}\sqrt{N(t-1,t]}Z^{(j)}(t-1,t]\right)^2.$$

Now we define the following vectors:

$$n := (\sqrt{N(i_{N(t_{j-1})}, i_{N(t_{j-1})+1}]}, \ldots, \sqrt{N(i_{N(t_j)-1}, i_{N(t_j)}]})^T,$$

$$z := (\sqrt{Z^{(j)}(i_{N(t_{j-1})}, i_{N(t_{j-1})+1}]}, \ldots, \sqrt{Z^{(j)}(i_{N(t_j)-1}, i_{N(t_j)}]})^T.$$

Then the ML estimator of σ_j^2 can be represented by the following equation:

$$\hat{\sigma}_j^2 = \frac{1}{N(t_{j-1},t_j]}z^Tz - \frac{1}{N^2(t_{j-1},t_j]}z^Tnn^Tz = \frac{1}{N(t_{j-1},t_j]}z^TAz,$$

where $A := I_{N(t_{j-1},t_j]} - \frac{1}{N(t_{j-1},t_j]}nn^T$ is a symmetric idempotent matrix and its rank is $\operatorname{rank}(A) = \operatorname{tr}(A) = N(t_{j-1},t_j] - 1$. It is obvious that the conditional distribution of the random vector z is given by

$$z\,|\,n \sim \mathcal{N}(\mu_j n, \sigma_j^2 I_{N(t_{j-1},t_j]}).$$

From Rencher & Schaalje (2008), as the noncentrality parameter is $\alpha = \frac{1}{2}(\mu_j n)^T A(\mu_j n) = 0$, one can deduce that

$$\frac{N(t_{j-1},t_j]\hat{\sigma}_j^2}{\sigma_j^2}\,\Bigg|\, N(t_{j-1},t_j] = \frac{z^TAz}{\sigma_j^2}\,\Bigg|\, N(t_{j-1},t_j] \sim \chi^2(N(t_{j-1},t_j]-1).$$

Consequently, (3) holds. For the parameter μ_j, it is clear that

$$\frac{\mu_j - \hat{\mu}_j}{\sigma_j/\sqrt{N(t_{j-1},t_j]}}\,\Bigg|\, N(t_{j-1},t_j] \sim \mathcal{N}(0,1).$$

Therefore, since $\hat{\mu}_j$ and $\hat{\sigma}_j^2$ are independent conditional on $N(t_{j-1},t_j]$, we have

$$\frac{\mu_j - \hat{\mu}_j}{\hat{\sigma}_j}\,\Bigg|\, N(t_{j-1},t_j] \sim t(N(t_{j-1},t_j]-1).$$

Consequently, also (4) holds. For the discount rate, we consider the following hypothesis:

$$H_0 : k = k_*.$$

In this case, restricted estimators are the solution of following constrained optimization problem:

$$\begin{cases} \mathcal{L}(\theta) \to \max, \\ \text{s.t. } k = k_*. \end{cases}$$

It can be shown that the restricted estimator of σ_u^2 is given by the following equation:

$$\hat{\sigma}_{u*}^2 = \frac{1}{T}\sum_{t=1}^{T}(P_t - (1+k_*)P_{t-1} + d_t)^2.$$

The restricted residual is $e_* = p + d - (1+k_*)p_{-1}$ and the LR statistic is given by

$$\mathrm{LR}(k_*) = T\ln\left(1 + \frac{e_*^T e_* - e^T e}{e^T e}\right) = T\ln\left(1 + \frac{(\hat{k}-k_*)^2 p_{-1}^T p_{-1}}{e^T e}\right).$$

Since $\mathrm{plim}_{T\to\infty}\{\frac{(\hat{k}-k_*)^2 p_{-1}^T p_{-1}}{e^T e}\} = 0$, the following approximation holds:

$$\mathrm{LR}(k_*) \approx \frac{T(\hat{k}-k_*)^2 p_{-1}^T p_{-1}}{e^T e}, \quad T \to \infty.$$

If the hypothesis $H_0 : k = k_*$ is true, then $\hat{k} - k_* = \frac{1}{p_{-1}^T p_{-1}} p_{-1}^T u$. Thus, for the numerator of above approximation, we have

$$(k - k_*)^2 p_{-1}^T p_{-1} = \frac{u^T p_{-1} p_{-1}^T u}{p_{-1}^T p_{-1}}.$$

Let $\bar{M} := \frac{p_{-1}p_{-1}^T}{p_{-1}^T p_{-1}}$. Then $\bar{M}$ is symmetric, idempotent and $\mathrm{rank}(\bar{M}) = \mathrm{tr}(\bar{M}) = 1$. As $\hat{\sigma}_u^2$ is the consistent estimator of σ_u^2, if the hypothesis $H_0 : k = k_*$ is true, then one can show that the following convergence holds:

$$\mathrm{LR}(k_*) = T\ln\left(1 + \frac{e_*^T e_* - e^T e}{e^T e}\right) = T\ln\left(1 + \frac{(\hat{k}-k_*)^2 p_{-1}^T p_{-1}}{e^T e}\right) \xrightarrow{d} \chi^2(1).$$

This completes the proof of the proposition. □

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